Thermodynamiques des espaces-temps de Margulis

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Remembering Tubabu

গাইব, মা, বীররসে ভাসি মহাগীত;

- মাইকেল মধুসূদন দত্ত, মেঘনাদবধ কাব্য।

Résumé

Dans ma thèse, je décris les feuilles stables et instables pour le flot géodésique sur l'espace des géodésiques non-errant de type espace d'un espace-temps de Margulis et je démontre des propriétés de contraction des feuilles sous le flot. Je montre aussi que la monodromie d'un espace-temps de Margulis est une représentation Anosov dans un groupe de Lie non semisimple. En outre, je montre que les applications limites de ces représentations Anosov et les reparamétrisations du flot géodésique initial. Enfin, à l'aide de la propriété métrique Anosov, nous définissons la métrique de pression sur l'espace modulaire des espaces-temps de Margulis sans pointes et je démontre qu'elle est définie positive sur les sections d'entropie constante.

Abstract

In my thesis I describe the stable and unstable leaves for the geodesic flow on the space of non-wandering spacelike geodesics of a Margulis Space Time and prove contraction properties of the leaves under the flow. I also show that monodromy of Margulis Space Times are “Anosov representations in non semi-simple Lie groups”. Moreover, I show that the limit maps and reparametrizations vary analytically. Finally using the metric Anosov property we define the Pressure metric on the Moduli Space of Margulis Space Times without “cusps” and show that it is positive definite on the constant entropy sections.

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ধন্যবাদজ্ঞাপন

প্রথমত আমি অধ্যাপক ফ্রঁসোয়া লাবুরিকে আমার আন্তরিক ধন্যবাদ জানাই তাঁর নির্দেশনা এবং আমার এই অভিসন্দর্ভের একজন অসাধারণ উপদেষ্টক হওয়ার জন্য। এই অভিসন্দর্ভের বিচারক অধ্যাপিকা আনা উইনহার্ড এবং অধ্যাপক থিয়েরি বার্বোকে তাদের গঠনমূলক পরামর্শের জন্য এবং অধ্যাপক ফ্রান্সিস বোনাহোঁ, অধ্যাপক সিলভাঁ ক্রোভিসিয়ে এবং অধ্যাপক ইঅ্যার মিনস্কিকে আমার অভিসন্দর্ভরক্ষণের পরীক্ষক হওয়ার জন্য আমার ধন্যবাদ। আমার সাথে বিভিন্ন সাহায্যকর অর্থপূর্ণ কথোপকথনের জন্য আমি ধন্যবাদ জানাই অধ্যাপক উইলিয়াম গোল্ডম্যান, অধ্যাপক মার্ক পলিকট, অধ্যাপক রিচার্ড ক্যানারি, অধ্যাপক অলিভিয়ে গীষার্দ, অধ্যাপক মার্টিন ব্রিজম্যান, অধ্যাপক ফ্রেদেরিক পলাঁ, ডঃ নাদিমপল্লী সন্তোষ, ডঃ মারিয়া বিয়াত্রিস পোৎজেত্তি এবং বিশেষভাবে ডঃ আন্দ্রেস সাম্বারিনোকে। আমাকে আমার অভিসন্দর্ভের অংশবিশেষ ফরাসী ভাষায় অনুবাদে সাহায্য করার জন্য আমি ধন্যবাদ জানাই অধ্যাপিকা সিলভি রুয়েত, অধ্যাপক ফ্রেদেরিক পলাঁ, শ্রীমান অরিন্দম বিশ্বাস, শ্রীমান শৌনক দত্ত এবং মুখ্যত শ্রীমান চার স্যের্ত কে। এছাড়া অভিসন্দর্ভরক্ষণ পরবর্তী অনুষ্ঠান সংগঠিত করায় আমাকে সাহায্য করার জন্য আমি ধন্যবাদ জানাই শ্রীমান কাজল দাস, শ্রীমান অরিন্দম বিশ্বাস, শ্রীমান সুদর্শন শীন্ডে এবং বিশেষত শ্রীমান সৌমাংশু ভূষণ চক্রবর্তী এবং শ্রীমান চার স্যের্ত কে।

ইউরোপীয় গবেষণা পরিষদের [1] প্রতি আমার অশেষ ধন্যবাদ আমাকে গবেষণাকালে আর্থিক ভাবে সহায়তা করার জন্য এবং অর্সের জাক হাদামার্দ গ্রন্থাগারকে আমার হার্দিক অভিনন্দন গণিতচর্চার এমন অসাধারণ পরিমন্ডল উপহার দেওয়ার জন্য। অধ্যাপক উইলিয়াম গোল্ডম্যান, অধ্যাপক মার্ক পলিকট, অধ্যাপক রিচার্ড ক্যানারি, অধ্যাপক অলিভিয়ে গীষার্দ, অধ্যাপক মার্টিন ব্রিজম্যান এবং ডঃ মারিয়া বিয়াত্রিস পোৎজেত্তির সাথে মুখোমুখি কথোপকথনের সুযোগ করে দেওয়ার জন্য আমি ধন্যবাদ জানাই আর্হাসে প্রেশার মেট্রিকের উপর হওয়া কিউ.জি.এম. সম্মেলনের সংগঠকদের, ব্রাউনে ``একজটিক জিওমেট্রিক স্ট্রাকচার্স" এর উপর হওয়া আই.সি.ই.আর.এম. কর্মশালার সংগঠকদের, আর্বানা-শ্যাম্পেন এবং মিশিগানে হওয়া জি.ই.এ.আর. রিট্রিটের সংগঠকদের এবং এম.এস.আর.আই এ হওয়া ডি.এম.এস কার্যক্রমের সংগঠকদের। এছাড়া আমাকে দিল্লির জওহরলাল নেহেরু বিশ্ববিদ্যালয়ে দু মাসের জন্য আমন্ত্রণ করার জন্য অধ্যাপিকা ঋদ্ধি শাহকে আমি ধন্যবাদ জানাই। কার্যত জওহরলাল নেহেরু বিশ্ববিদ্যালয়ে থাকাকালীন ই আমি আমার অভিসন্দর্ভের প্রথম এবং অন্যতম প্রধান বৃহ্যভেদ করি এছাড়া অবশিষ্ট অন্য দুটি প্রধান বৃহ্যভেদ হয় আমার আর্হাসে এবং অর্সেতে থাকাকালীন।

আমাকে আমার গাণিতিক জীবনের বিভিন্ন সময়ে গুরুত্বপূর্ণ পথপ্রদর্শন করার জন্য শ্রী চিন্ময় চৌধুরী (শ্রদ্ধেয় সি.সি.স্যার), অধ্যাপক মহান মহারাজ এবং অধ্যাপক ফ্রঁসোয়া লাবুরির প্রতি আমি আমার কৃতজ্ঞতা জানাই। আমি ধন্যবাদ জানাই আমার দোসর সৌম্যা শুক্লা, বন্ধু অর্পণ ঘোষ ও সাগ্নিক সেন এবং আমার ভাই গৌরব ঘোষকে তাদের বৌদ্ধিক সাঙ্গত্যের জন্য। পরিশেষে সব সময় আমার পাশে থাকার জন্য এবং আর্থিক প্রতিবন্ধকতা সত্ত্বেও আমাকে যথাযত শিক্ষাদানের জন্য দৃঢ় সংকল্প এবং কঠোর পরিশ্রমের জন্য আমি আমার মা শ্রীমতি পূর্ণিমা ঘোষ এবং আমার বাবা শ্রী স্বপন ঘোষকে আমার আন্তরিক চিরকৃতজ্ঞতা জানাই।

এছাড়াও এলিমেন্টস নামক বইটি লেখার জন্য আলেকজেন্দ্রিয়ার ইউক্লিডের প্রতি আমি কৃতজ্ঞ।

[1]এই ফলাফল গুলিতে উপনীত হওয়ার পিছনের গবেষণা ইউরোপীয় গবেষণা পরিষদের অন্তর্গত ইউরোপীয় জনসমাজ এর সপ্তম পরিকাঠামো কার্যক্রমের অনুদান চুক্তি (এফ পি ৭/২০০৭-২০১৩)/ই আর সি থেকে আর্থিক সহায়তা লাভ করেছে।

Contents

Description en français **1**

Introduction **11**

1 Affine Geometry **15**

2 Hyperbolic Geometry **17**
2.1 The Hyperboloid model . . . 17
2.2 The Horocycles . . . 18
2.3 The neutral section and the limit sections . . . 18
2.4 Relation with the cross ratio . . . 19
2.5 Quotient surfaces . . . 20

3 Margulis Space Times **21**
3.1 Definition and Existence . . . 21
3.2 Margulis Space Times and Surfaces . . . 21
3.3 The Representation Variety . . . 23
3.4 The Margulis Invariant . . . 23

4 Metric Anosov Property **25**

5 Metric Anosov structure on Margulis Space Time **27**
5.1 Metric space structure . . . 27
5.2 The lamination and its lift . . . 31
5.3 Contraction Properties . . . 41
5.4 Metric Anosov structure on the quotient . . . 45

6 Anosov representations **47**
6.1 Pseudo-Parabolic subgroups . . . 47
6.2 Geometric Anosov structure . . . 48
6.3 Gromov geodesic flow . . . 49
6.4 Anosov structure . . . 50

7 Thermodynamical formalism **53**
7.1 Hölder flows . . . 53
7.1.1 Reparametrizations . . . 53
7.1.2 Livšic-cohomology . . . 53
7.1.3 Periods and measures . . . 54
7.1.4 Entropy and pressure . . . 55
7.2 Entropy and pressure for Anosov flows . . . 55
7.3 Intersection and renormalised intersection . . . 56
7.3.1 Intersection . . . 56
7.3.2 Renormalized intersection . . . 57
7.4 Variation of the pressure and the pressure form . . . 57
7.4.1 First and second derivatives . . . 57
7.4.2 The pressure form . . . 58

8 Deformation Theory **61**
8.1 Transverse Analyticity . . . 61
8.2 Analyticity of limit maps . . . 63
8.3 Analyticity of Reparametrizations . . . 67
8.4 Deformation of the cross ratio . . . 71

9 Properties of the Pressure metric **75**
9.1 The thermodynamic mapping . . . 75
9.2 The Pressure metric . . . 76
9.3 Vectors with Pressure norm zero . . . 76
9.4 Margulis Multiverse . . . 78
9.5 Riemannian metric on Margulis Multiverse . . . 80

Description en français

Un espace-temps de Margulis M est un quotient de l'espace affine $\mathbb{A}$ de dimension trois par un groupe libre, non-abélien Γ agissant par des transformations affines dont la partie linéaire est discrète.

Grigory Margulis a utilisé ces espaces, dans [27] et [28], comme des exemples pour répondre par la négative à la question suivante de Milnor.

Question 1. *Est-ce que le groupe fondamental d'une variété complète, plate et affine est virtuellement polycyclique?*

Si M est un espace-temps de Margulis, alors le groupe fondamental $\pi_1(\mathsf{M})$ ne contient aucune translation. Les résultats de Fried–Goldman et Mess, dans [15], [30], impliquent qu'une variété affine plate complète soit est un espace-temps de Margulis, soit admet comme groupe fondamental un groupe polycyclique.

Dans ma thèse, je ne considère que les espace-temps de Margulis dont la partie linéaire ne contient aucun élément parabolique, bien que par des résultats de Drumm, il existe des espace-temps de Margulis dont la partie linéaire contient des éléments paraboliques.

Fried–Goldman ont montré dans [15] qu'un conjugué de la partie linéaire de l'action affine du groupe fondamental de M est un sous-groupe de $\mathsf{SO}(2,1)$ dans $\mathsf{GL}(3,\mathbb{R})$. Par conséquent, les espace-temps de Margulis proviennent d'homomorphismes injectifs

$$\rho : \Gamma \longrightarrow \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$$

où Γ est un groupe libre non-abélien de rang fini.

Notons l'espace des homomorphismes injectifs d'un groupe libre Γ dans un groupe de Lie G par $\mathsf{Hom}(\Gamma, G)$ et l'espace de cocycles par $\mathsf{Z}^1(\mathsf{L}_\rho(\Gamma), \mathbb{R}^3)$. Notons l'espace des homomorphismes ρ dans $\mathsf{Hom}(\Gamma, \mathsf{G})$ tels que $\rho(\Gamma)$ agit proprement sur $\mathbb{A}$ et que $\mathsf{L}_\rho(\Gamma)$ est discret et ne contient aucun élément parabolique par $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ où $\mathsf{G} := \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$. Notons aussi l'espace des modules des espace-temps de Margulis par $\mathcal{M}$. On remarque que:

$$\mathcal{M} \cong \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G}) / \sim$$

où $\rho_1 \sim \rho_2$ si et seulement si ρ_1 est un conjugué de ρ_2 par un élément du groupe G. Soient $\mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ l'espace de toutes les représentations Schottky de Γ dans $\mathsf{SO}^0(2,1)$ et

$$\mathcal{T} := \mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1)) / \sim$$

où $\varrho_1 \sim \varrho_2$ si et seulement si ϱ_1 est un conjugué de ϱ_2 par un élément du groupe $\mathsf{SO}^0(2,1)$. Goldman–Labourie–Margulis ont montré dans [18] que:

Theorem 0.0.1. *[Goldman–Labourie–Margulis] L'espace des modules des espace-temps de Margulis $\mathcal{M}$ est une partie ouverte dans le fibré tangent de la variété analytique $\mathcal{T}$.*

En conséquence,

Proposition 0.0.2. *L'espace des modules des espace-temps de Margulis $\mathcal{M}$ est une variété analytique.*

Les classes de parallélisme des géodésiques de type temps de M peuvent être paramétrées par une surface hyperbolique complète Σ. Des travaux récents de Danciger–Guéritaud–Kassel dans [13] montrent que M est une fibration de $\mathbb{R}$ sur Σ dont les fibres sont des géodésiques type temps.

Les travaux antérieurs de Charette–Goldman–Jones dans [12], Goldman–Labourie–Margulis dans [18] et Goldman–Labourie dans [17] ont montré que la dynamique de M est étroitement liée à celle de Σ. Jones–Charette–Goldman ont montré dans [12] qu'elle existe des géodésiques bi-spirales dans M et qu'ils correspondent aux géodésiques bi-spirales de Σ. Goldman–Labourie ont montré dans [17] que les géodésiques non errantes de type espace de M correspondent aux géodésiques non errantes de Σ.

En effet, soient $\rho \in \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ et $\Sigma_\rho := \mathsf{L}_\rho(\Gamma)\backslash\mathbb{H}$ où $\mathsf{L}_\rho(\Gamma)$ la partie linéaire de $\rho(\Gamma)$ et $\mathbb{H}$ le modèle de l'hyperboloïde du plan hyperbolique. Également, soient $\mathsf{U}_{\mathrm{rec}}\Sigma_\rho$ et $\mathsf{U}_{\mathrm{rec}}\mathsf{M}_\rho$ les espaces des points non errantes du flot géodésique, respectivement sur $\mathsf{U}\Sigma_\rho$ et sur UM_ρ, soit $\mathsf{U}^\rho_{\mathrm{rec}}\mathbb{H}$ son relèvement dans $\mathsf{U}\mathbb{H}$ et soit $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ le relèvement de $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ dans $\mathsf{U}\mathbb{A}$, où $\mathsf{U}\mathbb{A}$ et $\mathsf{U}\mathbb{H}$ sont les fibrs tangents unitaires de $\mathbb{A}$ et de $\mathbb{H}$ respectivement.

Dans [18] Goldman–Labourie–Margulis démontre le théorème suivant:

Theorem 0.0.3. *[Goldman–Labourie–Margulis] Soit $\rho : \Gamma \to \mathsf{G}$ un homomorphisme donnant lieu à un espace-temps de Margulis dont la partie linéaire $\mathsf{L}_\rho(\Gamma)$ ne contient aucun élément parabolique. Alors il existe une application*

$$N_\rho : \mathsf{U}^\rho_{\mathrm{rec}}\mathbb{H} \longrightarrow \mathbb{A}$$

et une fonction hölderienne positive

$$f_\rho : \mathsf{U}^\rho_{\mathrm{rec}}\mathbb{H} \longrightarrow \mathbb{R}$$

telles que

1. *pour tout $\gamma \in \Gamma$ on a $f_\rho \circ \mathsf{L}(\rho(\gamma)) = f_\rho$,*
2. *pour tout $\gamma \in \Gamma$ on a $N_\rho \circ \mathsf{L}(\rho(\gamma)) = \rho(\gamma)N_\rho$, et*
3. *pour tout $g \in \mathsf{U}^\rho_{\mathrm{rec}}\mathbb{H}$ et pour tout $t \in \mathbb{R}$ on a*

$$N_\rho(\tilde{\phi}_t g) = N_\rho(g) + \left(\int_0^t f_\rho(\tilde{\phi}_s(g))ds \right) \nu(g)$$

où ν est la section neutrale.

On appelle N_ρ une *section neutralisée*. Maintenant, en utilisant l'existence d'une section neutralisée, Goldman–Labourie ont demontré que:

Theorem 0.0.4. *[Goldman–Labourie] Soit $\rho : \Gamma \to \mathsf{G}$ un homomorphisme donnant lieu à un espace-temps de Margulis tel que la partie linéaire $\mathsf{L}_\rho(\Gamma)$ ne contient aucun élément parabolique. Également, soient $\mathsf{U}_{\mathrm{rec}}\Sigma_{\mathsf{L}_\rho}$ et $\mathsf{U}_{\mathrm{rec}}\mathsf{M}_\rho$ définis comme ci-dessus. Maintenant, si N_ρ est une section neutralisée, alors il existe une application injective $\hat{\mathtt{N}}_\rho$ telle que le diagramme suivant commute,*

$$\begin{array}{ccc} \mathsf{U}^{\rho}_{\mathrm{rec}}\mathbb{H} & \xrightarrow{\mathtt{N}_\rho} & \mathsf{U}\mathbb{A} \\ \pi\Big\downarrow & & \Big\downarrow\pi \\ \mathsf{U}_{\mathrm{rec}}\Sigma_{\mathsf{L}_\rho} & \xrightarrow{\hat{\mathtt{N}}_\rho} & \mathsf{UM} \end{array}$$

où $\mathtt{N}_\rho := (N_\rho, \nu)$. En plus, $\hat{\mathtt{N}}_\rho$ est une homéomorphism hölderienne sur $\mathsf{U}_{\mathrm{rec}}\mathsf{M}_\rho$ qui est aussi une équivalence des orbites.

Dans ma thèse, je commence par rappeler quelques notions préliminaires afin de préparer le terrain pour décrire explicitement les laminations stables et instables pour le flot géodésiques Φ de $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$.

Definition 0.0.5. *Soit $(\mathcal{X}, d)$ un espace métrique. Une lamination $\mathcal{L}$ de $\mathcal{X}$ est une relation d'équivalence sur $\mathcal{X}$ telle que pour tout x dans $\mathcal{X}$ il existe un voisinage ouvert $\mathcal{U}_x$ de x dans $\mathcal{X}$, deux espaces topologiques $\mathcal{U}_1$ et $\mathcal{U}_2$ et un homéomorphisme $\mathfrak{f}_x$ de $\mathcal{U}_1 \times \mathcal{U}_2$ sur $\mathcal{U}_x$ vérifiant les propriétés suivantes,*

1. *pour tous w, z dans $\mathcal{U}_x \cap \mathcal{U}_y$ on a $p_2\left(\mathfrak{f}_x^{-1}(w)\right) = p_2\left(\mathfrak{f}_x^{-1}(z)\right)$ si et seulement si $p_2\left(\mathfrak{f}_y^{-1}(w)\right) = p_2\left(\mathfrak{f}_y^{-1}(z)\right)$, où p_2 est la projection de $\mathcal{U}_1 \times \mathcal{U}_2$ sur $\mathcal{U}_2$,*
2. *pour tous w, z dans $\mathcal{X}$, on a $w\mathcal{L}z$ si et seulement s'il existe une suite finie des points $w_1, w_2, .., w_n$ dans $\mathcal{X}$ avec $w_1 = w$ et $w_n = z$, telle que w_{i+1} est dans $\mathcal{U}_{w_i}$, où $\mathcal{U}_{w_i}$ est un voisinage de w_i et $p_2\left(\mathfrak{f}_{w_i}^{-1}(w_i)\right) = p_2\left(\mathfrak{f}_{w_i}^{-1}(w_{i+1})\right)$ pour tout i dans $\{1, 2, .., n-1\}$.*

On appelle un tel homéomorphisme $\mathfrak{f}_x$ une *carte* et les classes d'equivalence les *feuilles*. Une *plaque ouverte* dans la carte correspondant à $\mathfrak{f}_x$ est un ensemble de la forme $\mathfrak{f}_x(\mathcal{V}_1 \times \{x_2\})$ où $x = \mathfrak{f}_x(x_1, x_2)$ et $\mathcal{V}_1$ est un ouvert dans $\mathcal{U}_1$. La *topologie plaque* sur $\mathcal{L}_x$ est engendrée par des plaques ouvertes. Un voisinage plaque de x est un voisinage pour la topologie plaque sur $\mathcal{L}_x$.

Definition 0.0.6. *Une structure du produit local sur $\mathcal{X}$ est une paire de deux laminations $(\mathcal{L}_1, \mathcal{L}_2)$ vérifiant les propriétes suivantes: pour tout x dans $\mathcal{X}$, il existe deux voisinages plaques $\mathcal{U}_1$, $\mathcal{U}_2$ de x, respectivement dans $\mathcal{L}_1$, $\mathcal{L}_2$ et un homéomorphisme $\mathfrak{f}_x$ de $\mathcal{U}_1 \times \mathcal{U}_2$ sur un voisinage $\mathcal{W}_x$ de x, tel que $\mathfrak{f}_x$ definit une carte pour les laminations $\mathcal{L}_1$ et $\mathcal{L}_2$.*

Supposons que ψ_t est un flot sur $\mathcal{X}$. Une lamination $\mathcal{L}$ invariante sous le flot ψ_t est *transverse* au flot, si pour tout x dans $\mathcal{X}$, il existe un voisinage plaque $\mathcal{U}_x$ de x dans $\mathcal{L}_x$, un espace topologique $\mathcal{V}$, un ϵ positif et une homéomorphism $\mathfrak{f}_x$ de $\mathcal{U}_x \times \mathcal{V} \times (-\epsilon, \epsilon)$ sur un voisinage ouvert $\mathcal{W}_x$ de x dans $\mathcal{X}$ vérifiant les propriétés suivantes:

$$\psi_t(\mathfrak{f}_x(u, v, s)) = \mathfrak{f}_x(u, v, s + t)$$

pour u dans $\mathcal{U}_x$, v dans $\mathcal{V}$ et pour s, t dans l'intervalle $(-\epsilon, \epsilon)$. Soit $\mathcal{L}^{\cdot}$ une lamination transverse au flot ψ_t. Définissons une nouvelle lamination $\mathcal{L}^{\cdot,0}$, appellée la *lamination centrale*, obtenue à partir de $\mathcal{L}^{\cdot}$ comme suit: on dit que y, z dans $\mathcal{X}$ appartiennent à la même classe d'equivalence de $\mathcal{L}^{\cdot,0}$ s'il existe $t \in \mathbb{R}$ tel que $\psi_t y$ et z appartiennent à la même classe d'equivalence de $\mathcal{L}^{\cdot}$.

Definition 0.0.7. *Une lamination $\mathcal{L}$ invariante sous un flot ψ_t est contractée sous le flot si et seulement s'il existe un nombre réel T_0 tel que pour tout x dans $\mathcal{X}$, il existe une carte $\mathfrak{f}_x$ d'un voisinage ouvert W_x de x, et pour deux points arbitraires y, z dans W_x avec y, z appartenant à la même classe d'equivalence de $\mathcal{L}$, on a,*

$$d(\psi_t y, \psi_t z) < \frac{1}{2} d(y, z)$$

pour tout $t > T_0$.

Definition 0.0.8. *Un flot ψ_t sur un espace métrique compact est un flot métrique Anosov, si et seulement s'il existe deux laminations $\mathcal{L}^+$ et $\mathcal{L}^-$ de $\mathcal{X}$ telles que les conditions suivantes soient satisfaites:*

1. *$(\mathcal{L}^+, \mathcal{L}^{-,0})$ définit une structure de produit locale sur $\mathcal{X}$,*
2. *$(\mathcal{L}^-, \mathcal{L}^{+,0})$ définit une structure de produit locale sur $\mathcal{X}$,*
3. *les feuilles de $\mathcal{L}^+$ sont contractées par le flot,*
4. *les feuilles de $\mathcal{L}^-$ sont contractées par le flot inverse.*

Dans un tel cas, $\mathcal{L}^+$, $\mathcal{L}^-$, $\mathcal{L}^{+,0}$ et $\mathcal{L}^{-,0}$ sont appelés respectivement les laminations stables, instables, stables centrales and instables centrales.

Je montre alors le résultat suivant:

Lemma 0.0.9. *Il existe une métrique sur $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ qui est localement équivalente de manière bilipschitz à une métrique sur $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ obtenue à partir de la restriction de n'importe quelle métrique euclidienne sur $\mathsf{T}\mathbb{A} \cong \mathbb{A} \times \mathbb{V}$ où $\mathbb{V}$ l'espace vectoriel correspondant à l'espace affine $\mathbb{A}$ et $\mathsf{T}\mathbb{A}$ le fibré tangent de $\mathbb{A}$.*

Soit v un vecteur de type espace de norme un et soient v^+ et v^- deux vecteurs dans le cône de lumière futur tels que $\det[v^-, v, v^+] > 0$. Alors, je définis

Definition 0.0.10. *Les partitions positives de $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ sont respectivement données par,*

$$\mathcal{L}^+_{(X,v)} := \tilde{\mathcal{L}}^+_{(X,v)} \cap \mathsf{U}_{\mathrm{rec}}\mathbb{A}$$

où $(X, v) \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ et

$$\tilde{\mathcal{L}}^+_{(X,v)} := \{(X + s_1 v^+, v + s_2 v^+) \mid s_1, s_2 \in \mathbb{R}\}.$$

Definition 0.0.11. *Les partitions négatives de $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ sont respectivement données par,*

$$\mathcal{L}^-_{(X,v)} := \tilde{\mathcal{L}}^-_{(X,v)} \cap \mathsf{U}_{\mathrm{rec}}\mathbb{A}$$

où $(X, v) \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ et

$$\tilde{\mathcal{L}}^-_{(X,v)} := \{(X + s_1 v^-, v + s_2 v^-) \mid s_1, s_2 \in \mathbb{R}\}.$$

Proposition 0.0.12. *Les partitions* $\underline{\mathcal{L}}^+$ *et* $\underline{\mathcal{L}}^-$ *décrivent deux laminations sur* $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$.

Je prouve aussi que:

Theorem 0.0.13. *Les laminations* $(\mathcal{L}^+, \mathcal{L}^{-,0})$ *et* $(\mathcal{L}^-, \mathcal{L}^{+,0})$ *définissent une structure de produit locale sur* $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$.

En fait, les laminations sont équivariantes sous l'action de Γ et sous le flot de géodésique.

Definition 0.0.14. *Les projections de* $\mathcal{L}^+$ *et* $\mathcal{L}^-$ *sur l'espace* $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ *sont notées respectivement par* $\underline{\mathcal{L}}^+$ *et* $\underline{\mathcal{L}}^-$, *où* $\mathcal{L}^+, \mathcal{L}^-$ *sont définis comme ci-dessus.*

Ensuite, je démontre le résultat suivant:

Theorem 0.0.15. *Soient* $\underline{\mathcal{L}}^+$ *et* $\underline{\mathcal{L}}^-$ *deux laminations de l'espace métrique* $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ *telles qu'elles sont définies dans la définition ci-dessus. Le flot géodésique sur l'espace des géodésiques non errantes de type espace dans* M *contracte exponentiellement la lamination* $\underline{\mathcal{L}}^+$ *par le flot positif et contracte exponentiellement la lamination* $\underline{\mathcal{L}}^-$ *par le flot negatif.*

Il s'en suit que $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ a une structure métrique Anosov.

En outre, dans cette thèse je définis la notion d e représentation Anosov dans le contexte du groupe de Lie non semisimple $\mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$. La notion d'une représentation Anosov d'un groupe discret dans un groupe de transformations G a été introduite par Labourie dans [25]. En suite, Guichard–Wienhard ont étudié les représentations Anosov dans les groupes de Lie semisimples en plus de détail dans [21]. Récemment, dans [8] Bridgeman–Canary–Labourie–Sambarino ont introduite le flot géodésique d'une représentation Anosov et le formalisme thérmodynamique dans ce cas, encore dans le contexte de G étant un groupe de Lie semisimple. Dans cette thèse, j'étudie des cas spéciaux et de nouveaux exemples de représentations Anosov lorsque G est le groupe non semisimple $\mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$.

Soit $\mathbb{X}$ l'espace de tous les plans affines nulles. On observe que G agit transitivement sur $\mathbb{X}$. Par conséquent, pour tout $P \in \mathbb{X}$, on a

$$\mathbb{X} = \mathsf{G}.P \cong \mathsf{G}/\mathsf{Stab}_{\mathsf{G}}(P).$$

Definition 0.0.16. *Si* $P \in \mathbb{X}$, *je définis*

$$\mathsf{P}_P := \mathsf{Stab}_{\mathsf{G}}(P).$$

J'appelle P_P *un sous-groupe pseudo-parabolique de* G.

Soit P_{X,w_1,w_2} le plan passant par un point $X \in \mathbb{A}$ avec l'espace vectoriel correspondant engendré par les vecteurs w_1 et w_2. On fixe un point $O \in \mathbb{A}$ et definit:

$$\mathsf{P}^{\pm} := \mathsf{Stab}_{\mathsf{G}}(P_{O,v_0,v_0^{\pm}}).$$

Aussi soit $\mathsf{L} = \mathsf{P}^+ \cap \mathsf{P}^-$. Notons la boundary de Gromov du groupe Γ par $\partial_\infty\Gamma$ et le flot géodesique de Gromov sur

$$\mathsf{U}_0\Gamma := \Gamma\backslash(\partial_\infty\Gamma^{(2)} \times \mathbb{R})$$

par ψ.

Definition 0.0.17. *On dit que ρ dans $\mathsf{Hom}(\Gamma, \mathsf{G})$ est $(\mathsf{G}, \mathsf{P}^{\pm})$-Anosov si et seulement s'il existe deux applications continues*

$$\xi_\rho^{\pm} : \partial_\infty\Gamma \longrightarrow \mathsf{G}/\mathsf{P}^{\pm}$$

telles que les suivantes sont vraies:

1. *Pour tout γ dans Γ, on a $\xi_\rho^{\pm} \circ \gamma = \rho(\gamma).\xi_\rho^{\pm}$.*
2. *Si $x \neq y$ dans $\partial_\infty\Gamma$, alors $(\xi_\rho^+(x), \xi_\rho^-(y))$ réside dans G/L.*
3. *Le fibré induit $\Xi_\rho^+ := (\xi_\rho^+ \circ \pi_1)^*\mathsf{E}^+$ est contracté par le révetement du flot $\tilde{\psi}_t$ quand $t \to \infty$, et le fibré induit $\Xi_\rho^- := (\xi_\rho^- \circ \pi_2)^*\mathsf{E}^-$ est contracté par le révetement du flot $\tilde{\psi}_t$ quand $t \to -\infty$.*

Les applications $\xi_\rho^{\pm}$ sont les *applications limites* associées à la representation $(\mathsf{G}, \mathsf{P}^{\pm})$-Anosov ρ.

Proposition 0.0.18. *Si ρ est dans $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$, alors ρ est $(\mathsf{G}, \mathsf{P}^{\pm})$-Anosov.*

En d'autres termes, les monodromies de l'espace-temps de Margulis sont "des représentations Anosov dans le groupe de Lie non semi-simple $\mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$ ".

J'utilise cette propriété Anosov et la théorie du formalisme thérmodynamique qui est développée par Bowen, Bowen–Ruelle, Parry–Pollicott, Pollicott et Ruelle et d'autres dans [5], [6], [33], [34], [35] pour définir l'entropie et l'intersection. En outre, j'utilise la propriété métrique Anosov pour montrer que l'entropie et l'intersection varient analytiquement sur $\mathcal{M}$. Enfin, je définis et j'étudie la métrique de pression sur $\mathcal{M}$.

L'étude de la métrique de pression dans le cadre des variétés de représentation a été initiée par McMullen et Bridgeman–Taylor respectivement dans [29], [9]. McMullen a donné la métrique de Weil–Petersson en termes de métrique de pression sur l'espace de Teichmüller. Bridgeman–Taylor ont généralisé le résultat au cas quasi-fuchsien dans [9]. Bridgeman a également étudié la métrique de pression dans le cadre du groupe de Lie semi-simple $\mathsf{SL}(2, \mathbb{C})$ dans [7].

Les résultats récents de Bridgeman–Canary–Labourie–Sambarino dans [8] prolongent cela dans le contexte de n'importe quel groupe de Lie semi-simple. Dans cette thèse, j'étudie le cas où le groupe de Lie en question est $\mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$, un groupe de Lie non semi-simple.

Soit

$$\rho : \Gamma \to \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$$

une représentation donnant lieu à un espace-temps de Margulis et soit $\alpha_\rho(\gamma)$ l'invariant de Margulis de $\gamma \in \Gamma$ pour la représentation ρ. Maintenant, pour un nombre réel positif T, soit

$$R_T(\rho) := \{\gamma \in \mathsf{O} \mid \alpha_\rho(\gamma) \leqslant T\}$$

où O est la collection de toutes les classes de conjugaison d'éléments de Γ . Nous définissons l'entropie comme suit:

Definition 0.0.19. *L'entropie de la représentations ρ est donnée par:*

$$h_\rho = \lim_{T\to\infty} \frac{1}{T} \log\left(\#R_T(\rho)\right).$$

Je démontre que $R_T(\rho)$ est de cardinal fini et h_ρ est bien défini, fini et positif.

Definition 0.0.20. *L'intersection de deux représentations ρ_1, ρ_2 est donnée par:*

$$I(\rho_1,\rho_2) = \lim_{T\to\infty} \frac{1}{\#R_T(\rho_1)} \sum_{[\gamma]\in R_T(\rho_1)} \frac{\alpha_{\rho_2}(\gamma)}{\alpha_{\rho_1}(\gamma)}.$$

Definition 0.0.21. *L'intersection renormalisée de deux représentations ρ_1 et ρ_2 est donnée par:*

$$J_{\rho_1}(\rho_2) = I(\rho_1,\rho_2)\frac{h_{\rho_2}}{h_{\rho_1}}.$$

Je démontre que l'intersection et l'intersection renormalisée sont bien définies et je montre aussi la proposition suivante:

Proposition 0.0.22. *Les applications suivantes sont analytiques réelles:*

$$h : \mathcal{M} \to \mathbb{R}$$

$$I : \mathcal{M}\times\mathcal{M} \to \mathbb{R}$$

$$J : \mathcal{M}\times\mathcal{M} \to \mathbb{R}$$

Afin de montrer que l'entropie et l'intersection varie analytiquement je prouve les deux résultats téchniques suivantes:

Theorem 0.0.23. *Soit $\{\rho_u\}_{u\in\mathcal{D}}$ une famille analytique réelle dans $\mathsf{Hom}(\Gamma,\mathsf{G})$ parametrisée par une disque $\mathcal{D}$ centrée en 0. Si ρ_0 est $(\mathsf{G},\mathsf{P}^\pm)$-Anosov avec des applications limites*

$$\xi_0^\pm : \partial_\infty\Gamma \longrightarrow \mathsf{G}/\mathsf{P}^\pm$$

alors il existe une sous-disque $\mathcal{D}_0$ de $\mathcal{D}$ (contenant 0), un positif μ et une application continue

$$\xi^+ : \mathcal{D}_0\times\partial_\infty\Gamma \longrightarrow \mathsf{G}/\mathsf{P}^+$$

avec les propriétes suivantes:

1. *Si u est dans $\mathcal{D}_0$, alors ρ_u est une représentation $(\mathsf{G},\mathsf{P}^\pm)$-Anosov avec l'application limite μ-Hölder donnée par*

$$\begin{aligned} \xi_u^+ : \partial_\infty\Gamma &\longrightarrow \mathsf{G}/\mathsf{P}^+ \\ x &\longmapsto \xi^+(u,x), \end{aligned}$$

2. *Si x est dans $\partial_\infty\Gamma$, alors l'application suivante est analytique réelle*

$$\begin{aligned} \xi_x^+ : \mathcal{D}_0 &\longrightarrow \mathsf{G}/\mathsf{P}^+ \\ u &\longmapsto \xi^+(u,x), \end{aligned}$$

3. *L'application de $\partial_\infty\Gamma$ à $\mathcal{C}^\omega(\mathcal{D}_0, \mathsf{G}/\mathsf{P}^+)$ donnée par $x \mapsto \xi^+_x$ est μ-hölderienne,*

4. *L'application de $\mathcal{D}_0$ à $\mathcal{C}^\mu(\partial_\infty\Gamma, \mathsf{G}/\mathsf{P}^+)$ donnée par $u \mapsto \xi^+_u$ est analytique réelle.*

Il s'ensuit des resultats de Goldman–Labourie–Margulis et de Goldman–Labourie respectivement dans [18] et dans [17] qu'il existe un homéomorphisme qui est aussi une équivalence des orbites entre $\mathsf{U}_0\Gamma$ et $\mathsf{U}_{\mathrm{rec}}\mathsf{M}_\rho$ tel que le flot affine linéaire sur $\mathsf{U}_{\mathrm{rec}}\mathsf{M}_\rho$ est une réparametrisation hölderienne du flot de Gromov. Donc pour tout $\rho \in \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ on a une application hölderienne, positive

$$\mathfrak{f}_\rho : \mathsf{U}_0\Gamma \to \mathbb{R}$$

qui donne la réparametrisation. En plus, notons que pour tout $\gamma \in \Gamma$, on a

$$\int_\gamma \mathfrak{f}_\rho = \alpha_\rho(\gamma)$$

où $\alpha_\rho(\gamma)$ est l'invariante de Margulis. Dans ma thèse je démontre que:

Proposition 0.0.24. *Soit $\{\rho_u\}_{u\in\mathcal{D}}$ une famille des homomorphismes analytiques réels où $\rho_u \in \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ est parameterisée par une disque $\mathcal{D}$ centrée en 0. Alors, il existe une sous-disque $\mathcal{D}_1$ centrée en 0 et une famille analytique réelle*

$$\{\mathsf{f}_u : \mathsf{U}_0\Gamma \to \mathbb{R}\}_{u\in\mathcal{D}_1}$$

de fonctions hölderienne, positives telles que la fonction f_u est cohomologue de Liv̆sic à la fonction $\mathfrak{f}_{\rho_u}$.

Enfin, je définis la métrique de pression comme étant la Hessienne de J, c'est-à-dire,

Definition 0.0.25. *Soit $\rho \in \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ et soient $v, w \in \mathsf{T}_\rho\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$. La métrique de pression est définie comme*

$$\mathsf{P}_\rho(v, w) := \mathsf{D}^2_\rho J_\rho(v, w).$$

Il résulte du formalisme thérmodynamique que la métrique de pression P sur $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ est definie non-negative. Je continue en démontrant:

Proposition 0.0.26. *Soit $\{\rho_t\}$ un chemin lisse dans $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ avec $\frac{d}{dt}\big|_{t=0} \rho_t = v$. Si $\mathsf{P}_\rho(v, v) = 0$ et $\frac{d}{dt}\big|_{t=0} h_{\rho_t} = 0$, alors pour tout γ dans Γ*

$$\left.\frac{d}{dt}\right|_{t=0} \alpha_{\rho_t}(\gamma) = 0.$$

Bridgeman–Canary–Labourie–Sambarino ont démontré le résultat suivant dans [8].

Proposition 0.0.27. *[Bridgeman, Canary, Labourie, Sambarino] Soit ϱ un point dans $\mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ où $\mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ est l'espace de toutes les représentations Schottky. Alors,*

$$\lim_{n\to\infty} \left(\ell_\varrho(\gamma^n\eta^n) - \ell_\varrho(\gamma^n) - \ell_\varrho(\eta^n)\right) = \log \mathsf{b}_\varrho(\eta^-, \gamma^-, \gamma^+, \eta^+)$$

où $\ell_\rho(\gamma)$ est la longueur de la géodesique fèrmée correspondant à $\varrho(\gamma)$.

En outre, dans [19] (voir aussi [16]) Goldman–Margulis ont démontré que:

Theorem 0.0.28. *[Goldman–Margulis] Soit $\{\varrho_t\}_{t\in(-1,1)} \subset \mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ un chemin lisse. Alors pour toutes $\gamma \in \Gamma$, on a*

$$\left.\frac{d}{dt}\right|_{t=0} \ell_{\varrho_t}(\gamma) = \alpha_{(\varrho_0,\dot{\varrho}_0)}(\gamma)$$

où $\ell_{\varrho_t}(\gamma)$ est la longueur de la géodesique fèrmée de Σ_{ϱ_t} correspondant à $\varrho_t(\gamma) \in \varrho_t(\Gamma)$ et $\dot{\varrho}_0 := \left.\frac{d}{dt}\right|_{t=0} \varrho_t$.

J'utilise l'intuition acquise dans les deux théorèmes précédents pour démontrer ce qui suit:

Proposition 0.0.29. *Soient $\{\rho_t\}_{t\in(-1,1)}$ un chemin lisse dans $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ et $X_{\rho_t(\gamma)}$ un point quelconque sur la seule droite affine fixée par $\rho_t(\gamma)$ où $\gamma \in \Gamma$. Alors, pour touts γ, η dans Γ, on a*

$$\begin{aligned}
&\lim_{n\to\infty} \left(\alpha_{\rho_t}(\gamma^n\eta^n) - \alpha_{\rho_t}(\gamma^n) - \alpha_{\rho_t}(\eta^n)\right)\\
&= \left\langle X_{\rho_t(\gamma)} - X_{\rho_t(\eta)} \mid \nu_{\rho_t}\left(\eta^-, \gamma^+\right) + \nu_{\rho_t}\left(\eta^+, \gamma^-\right)\right\rangle,\\
&\left.\frac{d}{dt}\right|_{t=0} \left\langle X_{\rho_t(\gamma)} - X_{\rho_t(\eta)} \mid \nu_{\rho_t}\left(\eta^-, \gamma^+\right) + \nu_{\rho_t}\left(\eta^+, \gamma^-\right)\right\rangle\\
&= \lim_{n\to\infty} \left.\frac{d}{dt}\right|_{t=0} \left(\alpha_{\rho_t}(\gamma^n\eta^n) - \alpha_{\rho_t}(\gamma^n) - \alpha_{\rho_t}(\eta^n)\right)
\end{aligned}$$

où ν est la section neutrale et $\langle|\rangle$ est la métrique Lorentzienne standarde sur $\mathbb{R}^3$.

Theorem 0.0.30. *Soit $\{\varrho_t\}_{t\in(-1,1)}$ un chemin lisse dans $\mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ tel que $\rho := (\varrho_0, \dot{\varrho}_0) \in \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ oú $\dot{\varrho}_0 := \left.\frac{d}{dt}\right|_{t=0} \varrho_t$. Alors on a*

$$\begin{aligned}
&\langle X_{\rho(\gamma)} - X_{\rho(\eta)} \mid \nu_\rho(\eta^-, \gamma^+) + \nu_\rho(\eta^+, \gamma^-)\rangle\\
&\quad = \left.\frac{d}{dt}\right|_{t=0} \log \mathsf{b}_{\varrho_t}(\eta^-, \gamma^-, \gamma^+, \eta^+)
\end{aligned}$$

où $X_{\rho(\gamma)}$ est un point quelconque sur la seule droite affine fixée par $\rho(\gamma)$ et $X_{\rho(\eta)}$ est un point quelconque sur la seule droite affine fixée par $\rho(\eta)$.

En plus, en utilisant les résultats ci-dessus, je montre que:

Lemma 0.0.31. *Si pour tout $\gamma \in \Gamma$ on a $\left.\frac{d}{dt}\right|_{t=0} \alpha_{\rho_t}(\gamma) = 0$ alors pour touts $\gamma, \eta \in \Gamma$ on a*

$$\left.\frac{d}{dt}\right|_{t=0} \mathsf{b}_{\rho_t}\left(\eta^+, \gamma^-, \gamma^+, \eta^-\right) = 0.$$

Et la proposition suivant s'ensuit:

Proposition 0.0.32. *Soit* $\{\rho_t\}_{t\in(-1,1)}$ *un chemin lisse dans* $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ *avec*

$$\left.\frac{d}{dt}\right|_{t=0} \rho_t = \dot{\rho}_0.$$

Si $\mathtt{P}_{\rho_0}(\dot{\rho}_0, \dot{\rho}_0) = 0$ *et* $\left.\frac{d}{dt}\right|_{t=0} h_{\rho_t} = 0$ *alors*

$$[\dot{\rho}_0] = 0$$

dans $\mathsf{H}^1_{\rho_0}(\Gamma, \mathfrak{g})$ *où* $\mathfrak{g}$ *est l'algèbre de Lie du groupe de Lie* G *et* $\mathsf{H}^1_{\rho_0}(\Gamma, \mathfrak{g})$ *est la cohomologie de groupe.*

Soit h_ρ l'entropie topologique qui est liée à une représentation $\rho \in \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$. Ensuite, je définis les *sections d'entropie constante* de $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ pour tout $k > 0$ comme:

$$\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_k := \{\rho \in \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G}) \mid h_\rho = k\}. \tag{0.0.1}$$

Notons que, si $(\varrho, \mathtt{u})$ est dans $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3) = \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ alors $(\varrho, c\mathtt{u})$ l'est aussi où $c > 0$.

Lemma 0.0.33. *Soit* $(\varrho, \mathtt{u}) \in \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3)$ *alors pour* $c > 0$, *on a*

$$h_{(\varrho, c\mathtt{u})} = \frac{1}{c}\, h_{(\varrho, \mathtt{u})}\,.$$

Proposition 0.0.34. *L'espace* $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_k$ *est une sous-variété analytique de codimension un dans* $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ *pour tout* $k > 0$.

Remark 0.0.35. *L'application suivante:*

$$\begin{aligned} \mathcal{I}_k : \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_1 &\longrightarrow \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_k \\ (\varrho, \mathtt{u}) &\longmapsto \left(\varrho, \frac{1}{k}\mathtt{u}\right) \end{aligned}$$

donne un isomorphisme analytique entre $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_1$ *et* $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_k$.

Proposition 0.0.36. *L'espace* $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ *est isomorphe analytiquement à l'espace produit* $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_1 \times \mathbb{R}$.

Definition 0.0.37. *Le multivers de Margulis avec entropie* k *est*

$$\mathcal{M}_k := \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_k/\sim$$

où $k > 0$ *et* $\rho_1 \sim \rho_2$ *si et seulement si* ρ_1 *est un conjugué de* ρ_2 *par un élément du groupe* $\mathsf{G} = \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$.

Enfin, je démontre que:

Theorem 0.0.38. *Soient* $\mathcal{M}_k$ *une section d'entropie constante de la variété analytique* $\mathcal{M}$ *avec entropie* k *et* $\mathtt{P}$ *la métrique de pression sur* $\mathcal{M}$. *Alors* $(\mathcal{M}_k, \mathtt{P}|_{\mathcal{M}_k})$ *est une variété riemannienne analytique.*

Et je conclus ma thèse en montrant le résultat suivant:

Theorem 0.0.39. *La signature de la métrique de pression* $\mathtt{P}$ *sur l'espace des modules* $\mathcal{M}$ *est* $(\dim(\mathcal{M}) - 1, 0)$.

Introduction

A Margulis Space Time M is a quotient of the three dimensional affine space by a free, non-abelian group acting as affine transformations with discrete linear part. Grigory Margulis used these spaces, in [27] and [28], as examples to answer Milnor's following question in the negative.

Question 2. *Is the fundamental group of a complete, flat, affine manifold virtually polycyclic? [31]*

If M is a Margulis Space Time then the fundamental group $\pi_1(\mathsf{M})$ does not contain any translation. By combining results of Fried–Goldman and Mess from [15], [30], a complete flat affine manifold either has a polycyclic fundamental group or is a Margulis Space Time. In this thesis I will only consider Margulis Space Times whose linear part contains no parabolic elements, although by Drumm there exist Margulis Space Times whose linear part contain parabolic elements.

Fried–Goldman showed in [15] that a conjugate of the linear part of the affine action of the fundamental group forms a subgroup of $\mathsf{SO}(2,1)$ in $\mathsf{GL}(3,\mathbb{R})$. Hence, Margulis Space Times arise from the injective homomorphisms

$$\rho : \Gamma \longrightarrow \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$$

where Γ is a non-abelian free group with finitely many generators. Goldman–Labourie–Margulis show in [18] that $\mathcal{M}$, the Moduli Space of Margulis Space Times, is an open subset of the representation variety. Therefore $\mathcal{M}$ is an analytic manifold.

The parallelism classes of timelike geodesics of M can be parametrized by a non-compact complete hyperbolic surface Σ. Recent work by Danciger–Guéritaud–Kassel in [13] have shown that M is a $\mathbb{R}$-bundle over Σ and the fibers are time like geodesics.

Previous works of Charette–Goldman–Jones in [12], Goldman–Labourie–Margulis in [18] and Goldman–Labourie in [17] showed that the dynamics of M is closely related to that of Σ. Jones–Charette–Goldman showed in [12] that bispiralling geodesics in M exists and they correspond to bispiralling geodesics in Σ. Goldman–Labourie showed in [17] that non-wandering spacelike geodesics in M correspond to non-wandering geodesics in Σ.

In this thesis, I first chalk out some preliminary notions, in order to prepare the ground to explicitly describe the stable and unstable laminations of $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$, the space of non-wandering spacelike geodesics in M, under the geodesic flow Φ. I carry on to show that the stable laminations get contracted by the forward flow and the unstable laminations get contracted by the backward flow. More precisely, I prove the following,

Theorem 0.0.40. *Let $\underline{\mathcal{L}}^+$ and $\underline{\mathcal{L}}^-$ be two laminations of the metric space $\mathsf{U}_{\text{rec}}\mathsf{M}$ as defined in definition 5.2.13. The geodesic flow on the space of non-wandering spacelike geodesics in M contracts $\underline{\mathcal{L}}^+$ exponentially in the forward direction of the flow and contracts $\underline{\mathcal{L}}^-$ exponentially in the backward direction of the flow.*

Hence it follows that $\mathsf{U}_{\text{rec}}\mathsf{M}$ has a metric Anosov structure.

Moreover, in this thesis I define the notion of an Anosov representation in the context of the non-semisimple Lie group $\mathsf{SO}^0(2,1)\ltimes\mathbb{R}^3$. The notion of an Anosov representation of a discrete group in a group G of transformations was first introduced by Labourie in [25]. Later, Guichard–Wienhard studied Anosov representations into semisimple Lie groups in more details in [21]. Recently, in [8] Bridgeman–Canary–Labourie–Sambarino introduce the geodesic flow of an Anosov representation and the thermodynamical formalism in this picture, again in the context of G being any semisimple Lie group. In this thesis I study special cases and new examples of Anosov representations when G is the non-semisimple Lie group $\mathsf{SO}^0(2,1)\ltimes\mathbb{R}^3$. Using this definition I carry on to prove the following theorem:

Theorem 0.0.41. *Let N be the space of all oriented space-like affine lines in the three dimensional affine space and let $\mathcal{L}$ be the orbit foliation of the flow Φ on $\mathsf{U}_{\text{rec}}\mathsf{M}$. Then there exist a pair of foliations on N so that $(\mathsf{U}_{\text{rec}}\mathsf{M},\mathcal{L})$ admits a geometric $(\mathsf{N},\mathsf{SO}^0(2,1)\ltimes\mathbb{R}^3)$ Anosov structure.*

In other words, monodromies of Margulis Space Times are "Anosov representations in the non semisimple Lie group $\mathsf{SO}^0(2,1)\ltimes\mathbb{R}^3$".

I use this Anosov property and the theory of thermodynamical formalism developed by Bowen, Bowen–Ruelle, Parry–Pollicott, Pollicott and Ruelle and others in [5], [6], [33], [34], [35] to define the entropy and intersection. Moreover, I use the metric Anosov property to show that the entropy and intersection vary analytically over $\mathcal{M}$. Finally, I define and study the Pressure metric on $\mathcal{M}$.

The study of Pressure metric in the context of representation varieties was started by McMullen and Bridgeman–Taylor respectively in [29], [9]. McMullen gave a Pressure metric formulation of the Weil–Petersson metric on the Teichmüller Space. Bridgeman–Taylor generalised the result to the quasi-Fuchsian case in [9]. Bridgeman also studied the Pressure metric in the context of the semisimple Lie group $\mathsf{SL}(2,\mathbb{C})$ in [7]. Recent results by Bridgeman–Canary–Labourie–Sambarino in [8] extend it in the context of any semisimple Lie group. In this thesis I study the case where the Lie group in question is $\mathsf{SO}^0(2,1)\ltimes\mathbb{R}^3$, a non-semisimple Lie group.

Let $\rho:\Gamma\to\mathsf{SO}^0(2,1)\ltimes\mathbb{R}^3$ be a representation giving rise to a Margulis Space Time and let $\alpha_\rho(\gamma)$ be the Margulis Invariant of $\gamma\in\Gamma$ for the representation ρ. Now for a positive real number T let

$$R_T(\rho):=\{\gamma\in\mathsf{O}\mid\alpha_\rho(\gamma)\leqslant T\}$$

where O is the collection of all conjugacy classes of elements of Γ. Let us define the entropy as follows:

$$h_\rho=\lim_{T\to\infty}\frac{1}{T}\log\left(\#R_T(\rho)\right).$$

I will show that $R_T(\rho)$ has finite cardinality and h_ρ is well defined, finite and positive (follows from theorem 0.0.40, theorem 3.2.1 and proposition 7.2.1). Moreover, let us define the intersection of two representations ρ_1, ρ_2 as

$$I(\rho_1, \rho_2) = \lim_{T\to\infty} \frac{1}{\#R_T(\rho_1)} \sum_{[\gamma]\in R_T(\rho_1)} \frac{\alpha_{\rho_2}(\gamma)}{\alpha_{\rho_1}(\gamma)},$$

and the renormalised intersection of the two representations ρ_1, ρ_2 as

$$J_{\rho_1}(\rho_2) = I(\rho_1, \rho_2)\frac{h_{\rho_2}}{h_{\rho_1}}.$$

I show that the intersection and the renormalised intersection are well defined (follows from theorem 0.0.40, theorem 3.2.1, equation 7.3.1). I also show that the maps h, I, J are analytic over the analytic manifold $\mathcal{M}$ (follows from proposition 7.4.5 and proposition 8.3.1). Finally, I define the pressure metric as the Hessian of J, that is,

$$\mathsf{P}(v, w) = \mathsf{D}^2_{[\rho]} J_{[\rho]}(v, w)$$

where $v, w \in \mathsf{T}_{[\rho]}\mathcal{M}$. And I prove the following two theorems:

Theorem 0.0.42. *Let $\mathcal{M}_k$ be a constant entropy section of the analytic manifold $\mathcal{M}$ with entropy k and let P be the Pressure metric on $\mathcal{M}$. Then $(\mathcal{M}_k, \mathsf{P}|_{\mathcal{M}_k})$ is an analytic Riemannian manifold.*

Theorem 0.0.43. *The Pressure metric P has signature $(\dim(\mathcal{M}) - 1, 0)$ over the moduli space $\mathcal{M}$.*

In the process I also obtain a new formula for the deformation of the cross ratio (theorem 8.4.4).

Finally, I would like to mention that section 6, 7, 9 and 10 of this thesis contain most of my original work.

Chapter 1

Affine Geometry

An *affine space* is a set $\mathbb{A}$ together with a vector space $\mathbb{V}$ and a faithful and transitive group action of $\mathbb{V}$ on $\mathbb{A}$. We call $\mathbb{V}$ the underlying vector space of $\mathbb{A}$ and refer to its elements as translations. An *affine transformation* F between two affine spaces $\mathbb{A}_1$ and $\mathbb{A}_2$, is a map such that for all x in $\mathbb{A}_1$ and for all v in $\mathbb{V}_1$, F satisfies the following property:

$$F(x+v) = F(x) + \mathtt{L}(F).v \tag{1.0.1}$$

for some linear transformation $\mathtt{L}(F)$ between $\mathbb{V}_1$ and $\mathbb{V}_2$. Therefore, by fixing an origin O in $\mathbb{A}$, one can represent an affine transformation F, from $\mathbb{A}$ to itself as a combination of a linear transformation and a translation. More precisely,

$$F(O+v) = O + \mathtt{L}(F).v + (F(O) - O)\,. \tag{1.0.2}$$

We denote $(F(O) - O)$ by $\mathtt{u}(F)$. Let us denote the space of affine automorphisms of $\mathbb{A}$ onto itself by $\mathsf{Aff}(\mathbb{A})$.
Let $\mathsf{GL}(\mathbb{V})$ be the general linear group of $\mathbb{V}$. We consider the semidirect product $\mathsf{GL}(\mathbb{V}) \ltimes \mathbb{V}$ of the two groups $\mathsf{GL}(\mathbb{V})$ and $\mathbb{V}$ where the multiplication is defined by

$$(g_1, v_1).(g_2, v_2) := (g_1 g_2, v_1 + g_1.v_2) \tag{1.0.3}$$

for g_1, g_2 in $\mathsf{GL}(\mathbb{V})$ and v_1, v_2 in $\mathbb{V}$. Using equation 1.0.2 we obtain that the following map:

$$F \mapsto (\mathtt{L}(F), \mathtt{u}(F))$$

defines an isomorphism between $\mathsf{Aff}(\mathbb{A})$ and $\mathsf{GL}(\mathbb{V}) \ltimes \mathbb{V}$.
Let us denote the tangent bundle of $\mathbb{A}$ by $\mathsf{T}\mathbb{A}$. The tangent bundle $\mathsf{T}\mathbb{A}$ of an affine space $\mathbb{A}$ is a trivial bundle and is canonically isomorphic to $\mathbb{A} \times \mathbb{V}$ as a bundle. The geodesic flow $\tilde{\Phi}$ on $\mathsf{T}\mathbb{A}$ is defined as follows,

$$\begin{aligned} \tilde{\Phi}_t \colon \mathsf{T}\mathbb{A} &\longrightarrow \mathsf{T}\mathbb{A} \\ (p, v) &\mapsto (p + tv, v). \end{aligned} \tag{1.0.4}$$

Chapter 2

Hyperbolic Geometry

2.1 The Hyperboloid model

Let $\left(\mathbb{R}^{2,1}, \langle|\rangle\right)$ be a Minkowski Space Time where the quadratic form corresponding to the metric $\langle|\rangle$ is given by

$$\mathcal{Q} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.1.1}$$

Let $\mathsf{SO}(2,1)$ denote the group of linear transformations of $\mathbb{R}^{2,1}$ preserving the metric $\langle|\rangle$ and $\mathsf{SO}^0(2,1)$ be the connected component containing the identity of $\mathsf{SO}(2,1)$.
The cross product $\boxtimes$ associated with this quadratic form is defined as follows:

$$u \boxtimes v := (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_2v_1 - u_1v_2)^t \tag{2.1.2}$$

where u, v is denoted by $(u_1, u_2, u_3)^t$ and $(v_1, v_2, v_3)^t$ respectively. The cross product $\boxtimes$ satisfies the following properties for all u, v in $\mathbb{R}^{2,1}$:

$$\begin{aligned} &\langle u, v \boxtimes w\rangle = \det[u, v, w], \\ &\langle u \boxtimes v, u \boxtimes v\rangle = \langle u, v\rangle^2 - \langle u, u\rangle\langle v, v\rangle, \\ &u \boxtimes v = -v \boxtimes u. \end{aligned} \tag{2.1.3}$$

Now for all real number k we define,

$$\mathsf{S}^k := \{v \in \mathbb{R} \mid \langle v, v\rangle \ = k\}.$$

We note that S^{-1} has two components. We denote the component containing $(0,0,1)^t$ as $\mathbb{H}$. The quadratic form gives rise to a Riemannian metric of constant negative curvature on the submanifold $\mathbb{H}$ of $\mathbb{R}^{2,1}$. The space $\mathbb{H}$ is called the *hyperboloid model of hyperbolic geometry.* Let $\mathsf{U}\mathbb{H}$ denote the unit tangent bundle of $\mathbb{H}$. The map

$$\begin{aligned} \Theta : \mathsf{SO}^0(2,1) &\longrightarrow \mathsf{U}\mathbb{H} \\ g &\longmapsto \left(g(0,0,1)^t, g(0,1,0)^t\right), \end{aligned} \tag{2.1.4}$$

gives an analytic identification between $\mathsf{SO}^0(2,1)$ and $\mathsf{U}\mathbb{H}$. Let $\tilde{\phi}_t$ denote the geodesic flow on $\mathsf{U}\mathbb{H} \cong \mathsf{SO}^0(2,1)$. We note that $\tilde{\phi}_t(g) = g.a(t)$ where

$$a(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(t) & \sinh(t) \\ 0 & \sinh(t) & \cosh(t) \end{pmatrix}. \tag{2.1.5}$$

We also note that $\tilde{\phi}_t$ is the image of the geodesic flow on $\mathsf{PSL}(2,\mathbb{R})$ under the identification of $\mathsf{PSL}(2,\mathbb{R})$ and $\mathsf{SO}^0(2,1)$.

2.2 The Horocycles

There is a canonical metric $d_{\mathsf{U}\mathbb{H}}$ on the unit tangent bundle $\mathsf{U}\mathbb{H}$ whose restriction on $\mathbb{H}$ is the hyperbolic metric. The metric $d_{\mathsf{U}\mathbb{H}}$ is unique up to the action of the maximal compact subgroup of $\mathsf{SO}^0(2,1)$. Let $g \in \mathsf{SO}^0(2,1) \cong \mathsf{U}\mathbb{H}$. We recall that the *horocycles* $\tilde{\mathcal{H}}^{\pm}_g$ for the geodesic flow $\tilde{\phi}$ passing through the point g is defined as follows:

$$\tilde{\mathcal{H}}^{+}_g := \{h \in \mathsf{U}\mathbb{H} \mid \lim_{t\to\infty} d_{\mathsf{U}\mathbb{H}}(\tilde{\phi}_t g, \tilde{\phi}_t h) = 0\}, \tag{2.2.1}$$

$$\tilde{\mathcal{H}}^{-}_g := \{h \in \mathsf{U}\mathbb{H} \mid \lim_{t\to-\infty} d_{\mathsf{U}\mathbb{H}}(\tilde{\phi}_t g, \tilde{\phi}_t h) = 0\}. \tag{2.2.2}$$

We note that under the identification Θ, the horocycle $\tilde{\mathcal{H}}^{\pm}_g$ passing through g is given by $g.u^{\pm}(t)$, where $u^{\pm}(t)$ are defined as follows:

$$u^{+}(t) := \begin{pmatrix} 1 & -2t & 2t \\ 2t & 1-2t^2 & 2t^2 \\ 2t & -2t^2 & 1+2t^2 \end{pmatrix}, \tag{2.2.3}$$

$$u^{-}(t) := \begin{pmatrix} 1 & 2t & 2t \\ -2t & 1-2t^2 & -2t^2 \\ 2t & 2t^2 & 1+2t^2 \end{pmatrix}. \tag{2.2.4}$$

We also note that $\tilde{\mathcal{H}}^{\pm}$ is the image of the horocycles of $\mathsf{PSL}(2,\mathbb{R})$ under the identification of $\mathsf{PSL}(2,\mathbb{R})$ and $\mathsf{SO}^0(2,1)$.

2.3 The neutral section and the limit sections

In this section we define the following important maps and describe their properties. Let ν be defined as follows:

$$\begin{aligned} \nu\colon \mathsf{SO}^0(2,1) &\longrightarrow \mathsf{S}^1 \\ g &\longmapsto g(1,0,0)^t, \end{aligned} \tag{2.3.1}$$

and also let $\nu^{\pm}$ be defined as follows:

$$\begin{aligned} \nu^{\pm}\colon \mathsf{SO}^0(2,1) &\longrightarrow \mathsf{S}^0 \\ g &\longmapsto g.\left(0, \pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^t. \end{aligned} \tag{2.3.2}$$

The map ν is called the *neutral section* and the maps ν^+(respectively ν^-) are called the *positive* (respectively *negative*) *limit sections.* We list a few properties of the neutral section and the limit sections as follows:

$$\nu(\tilde{\phi}_t g) = \nu(g), \tag{2.3.3}$$
$$\nu(h.g) = h.\nu(g), \tag{2.3.4}$$
$$\nu^{\pm}(\tilde{\phi}_t g) = e^{\pm t}\nu^{\pm}(g), \tag{2.3.5}$$
$$\nu^{\pm}(h.g) = h.\nu^{\pm}(g), \tag{2.3.6}$$
$$\nu^{+}(g.u^{+}(t)) = \nu^{+}(g), \tag{2.3.7}$$
$$\nu^{-}(g.u^{-}(t)) = \nu^{-}(g). \tag{2.3.8}$$

where $t \in \mathbb{R}$ and $g, h \in \mathsf{SO}^0(2,1)$.

2.4 Relation with the cross ratio

Let $\partial_\infty\mathbb{H}$ denote the boundary of $\mathbb{H}$. We recall that

$$\mathsf{U}\mathbb{H}/_\sim \cong \partial_\infty\mathbb{H} \times \partial_\infty\mathbb{H} \setminus \Delta \tag{2.4.1}$$

where $g \sim \tilde{\phi}_t(g)$ for all real number t and Δ denotes the diagonal of $\partial_\infty\mathbb{H} \times \partial_\infty\mathbb{H}$. Now from equation 2.3.3 we get that the neutral section is invariant under the geodesic flow on $\mathsf{U}\mathbb{H}$. As the neutral section is invariant under the geodesic flow it induces an analytic map,

$$\nu\colon \partial_\infty\mathbb{H} \times \partial_\infty\mathbb{H} \setminus \Delta \longrightarrow \mathsf{S}^1. \tag{2.4.2}$$

Let γ be a hyperbolic element of $\mathsf{SO}^0(2,1)$ acting on $\mathbb{H}$ and

$$\gamma^{\pm} := \lim_{n\to\pm\infty} \gamma^n x$$

where x is some point in $\mathbb{H}$. We recall that the definition of $\gamma^{\pm}$ is independent of the point x in $\mathbb{H}$. We notice that

$$\gamma\nu(\gamma^-, \gamma^+) = \nu(\gamma^-, \gamma^+), \tag{2.4.3}$$

that is, $\nu(\gamma^-, \gamma^+)$ is an eigenvector of γ with eigenvalue 1. Moreover for a, b, c, d in $\partial_\infty\mathbb{H}$ let

$$\mathsf{b}(a, b, c, d) := \frac{1}{2}\left(1 + \langle \nu(a, d) \mid \nu(b, c)\rangle\right). \tag{2.4.4}$$

Now we list a few identities satisfied by ν and b:

$$\nu(a, b) + \nu(b, a) = 0, \tag{2.4.5}$$
$$\langle \nu(a, b) \mid \nu(a, c)\rangle = 1, \tag{2.4.6}$$
$$\mathsf{b}(d, b, c, a)\nu(a, b) + \mathsf{b}(a, b, c, d)\nu(a, c) = \nu(a, d), \tag{2.4.7}$$
$$\mathsf{b}(a, b, c, d) = \mathsf{b}(b, a, d, c) = \mathsf{b}(d, c, b, a), \tag{2.4.8}$$
$$\mathsf{b}(a, b, c, d) + \mathsf{b}(d, b, c, a) = 1, \tag{2.4.9}$$
$$\mathsf{b}(a, w, c, d)\mathsf{b}(w, b, c, d) = \mathsf{b}(a, b, c, d). \tag{2.4.10}$$

We notice that b is the classical *cross ratio*.

2.5 Quotient surfaces

Let Γ be a free, nonabelian subgroup with finitely many generators. We consider the left action of Γ on $\mathsf{U}\mathbb{H}$. We notice that the action of Γ being from the left and the action of $a(t)$ being from the right, the two actions commute. Furthermore, given a free and proper action of Γ on $\mathsf{U}\mathbb{H}$, one gets an isomorphism between $\Gamma\backslash\mathsf{U}\mathbb{H}$ and $\mathsf{U}\Sigma$, where $\mathsf{U}\Sigma$ is the unit tangent bundle of the surface $\Sigma := \Gamma\backslash\mathbb{H}$. We note that the flow $\tilde{\phi}$ on $\mathsf{U}\mathbb{H}$ gives rise to a flow ϕ on $\mathsf{U}\Sigma$.

Let x_0 be a point in $\mathbb{H}$. Let $\Gamma.x_0$ denote the orbit of x_0 under the action of Γ. We denote the closure of $\Gamma.x_0$ inside the closure of $\mathbb{H}$ by $\overline{\Gamma.x_0}$. We define the *limit set* of the group Γ to be the space $\overline{\Gamma.x_0}\backslash\Gamma.x_0$ and denote it by $\Lambda_\infty\Gamma$. We note that the collection $\overline{\Gamma.x_0}\backslash\Gamma.x_0$ is independent of the particular choice of x_0. We also know that $\Lambda_\infty\Gamma$ is compact.

A point $g \in \mathsf{U}\Sigma$ is called a *wandering point* of the flow ϕ if and only if there exists an ϵ-neighborhood $\mathcal{B}_\epsilon(g) \subset \mathsf{U}\Sigma$ around g and a real number t_0 such that for all $t > t_0$ we have that

$$\mathcal{B}_\epsilon(g) \cap \phi_t\mathcal{B}_\epsilon(g) = \emptyset.$$

Moreover, a point is called *non-wandering* if and only if it is not a wandering point.

Let $\mathsf{U}_{\mathrm{rec}}\Sigma$ be the space of all non-wandering points of the geodesic flow ϕ on $\mathsf{U}\Sigma$. We denote the lift of the space $\mathsf{U}_{\mathrm{rec}}\Sigma$ in $\mathsf{U}\mathbb{H}$ by $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$. Now if the action of Γ on $\mathbb{H}$ is free and proper and moreover Γ contains no parabolics, then the space $\mathsf{U}_{\mathrm{rec}}\Sigma$ is compact. We note that the subspace $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$ can also be given an alternate description as follows:

$$\mathsf{U}_{\mathrm{rec}}\mathbb{H} = \left\{(x, v) \in \mathsf{U}\mathbb{H} \mid \lim_{t\to\pm\infty} \tilde{\phi}^1_t x \in \Lambda_\infty\Gamma\right\}$$

where $\tilde{\phi}_t(x, v) = (\tilde{\phi}^1_t x, \tilde{\phi}^2_t v)$. Furthermore, we note that the space $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$ can be identified with the space $(\Lambda_\infty\Gamma \times \Lambda_\infty\Gamma \setminus \{(x, x) \mid x \in \Lambda_\infty\Gamma\}) \times \mathbb{R}$.

Chapter 3

Margulis Space Times

3.1 Definition and Existence

A *Margulis Space Time* M is a quotient manifold of the three dimensional affine space $\mathbb{A}$ by a free, non-abelian group Γ which acts freely and properly as affine transformations with discrete linear part. In [27] and [28] Margulis showed the existence of these spaces. Later in [14] Drumm introduced the notion of *crooked planes* and constructed fundamental domains of a certain class of Margulis Space Times. In his construction the crooked planes give the boundary of appropriate fundamental domains for a certain class of Margulis Space Times. Recently, in [13] Danciger–Guéritaud–Kassel showed that for any Margulis Space Time one can find a fundamental domain whose boundaries are given by union of crooked planes.

If Γ_0 is a subgroup of $\mathsf{GL}(\mathbb{R}^3) \ltimes \mathbb{R}^3$ such that $\mathsf{M}_0 := \Gamma_0 \backslash \mathbb{A}$ is a Margulis Space Time then by a result proved by Fried–Goldman in [15] we get that a conjugate of $\mathtt{L}(\Gamma_0)$ is a subgroup of $\mathsf{SO}^0(2,1)$. Therefore without loss of generality we can denote a Margulis Space Time by a conjugacy class of homomorphisms

$$\rho : \Gamma \longrightarrow \mathsf{G} := \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$$

where Γ is a free non-abelian group with finitely many generators. In this thesis I will only consider Margulis Space Times $[\rho]$ such that $\mathtt{L}(\rho(\Gamma))$ contains no parabolic elements.

3.2 Margulis Space Times and Surfaces

Let $\mathsf{M}_\rho := \rho(\Gamma) \backslash \mathbb{A}$ be a Margulis Space Time such that $\mathtt{L}(\rho(\Gamma))$ contains no parabolic elements. Then the action of $\mathtt{L}(\rho(\Gamma))$ on $\mathbb{H}$ is Schottky. Hence $\Sigma_{\mathtt{L}_\rho} := \mathtt{L}(\rho(\Gamma)) \backslash \mathbb{H}$ is a non-compact surface with no cusps.

Now let TM_ρ be the tangent bundle of M_ρ. As $\mathtt{L}(\rho(\Gamma)) \subset \mathsf{SO}^0(2,1)$ we have that TM_ρ carries a Lorentzian metric $\langle | \rangle$. Let

$$\mathsf{UM}_\rho := \{(X, v) \in \mathsf{TM}_\rho \mid \langle v \mid v \rangle_X = 1\}.$$

We note that $\mathsf{UM}_\rho \cong \rho(\Gamma) \backslash \mathsf{U}\mathbb{A}$ where $\mathsf{U}\mathbb{A} := \mathbb{A} \times \mathsf{S}^1$. The geodesic flow $\tilde{\Phi}$ on $\mathsf{T}\mathbb{A}$ gives rise to a flow Φ on UM_ρ.

We recall that a point $(X,v) \in \mathsf{UM}_\rho$ is called a *wandering point* of the flow Φ if and only if there exists an ϵ-neighborhood $\mathcal{B}_\epsilon(X,v) \subset \mathsf{U}\Sigma$ around (X,v) and a real number t_0 such that for all $t > t_0$ we have that

$$\mathcal{B}_\epsilon(X,v) \cap \Phi_t \mathcal{B}_\epsilon(X,v) = \emptyset.$$

Moreover, a point is called *non-wandering* if and only if it is not a wandering point.

We denote the space of all non-wandering points of the flow Φ on UM_ρ by $\mathsf{U}_{\mathrm{rec}}\mathsf{M}_\rho$. Moreover, we denote the lift of $\mathsf{U}_{\mathrm{rec}}\mathsf{M}_\rho$ into $\mathsf{U}\mathbb{A}$ by $\mathsf{U}^\rho_{\mathrm{rec}}\mathbb{A}$.

In [18] Goldman–Labourie–Margulis proved the following theorem:

Theorem 3.2.1. *[Goldman–Labourie–Margulis] Let $\rho : \Gamma \to \mathsf{G}$ be a homomorphism giving rise to a Margulis Space Time and let $\mathtt{L}(\rho(\Gamma))$ contains no parabolic elements. Then there exists a map*

$$N_\rho : \mathsf{U}^\rho_{\mathrm{rec}}\mathbb{H} \longrightarrow \mathbb{A}$$

and a positive Hölder continuous function

$$f_\rho : \mathsf{U}^\rho_{\mathrm{rec}}\mathbb{H} \longrightarrow \mathbb{R}$$

such that

1. *for all $\gamma \in \Gamma$ we have $f_\rho \circ \mathtt{L}(\rho(\gamma)) = f_\rho$,*
2. *for all $\gamma \in \Gamma$ we have $N_\rho \circ \mathtt{L}(\rho(\gamma)) = \rho(\gamma) N_\rho$, and*
3. *for all $g \in \mathsf{U}^\rho_{\mathrm{rec}}\mathbb{H}$ and for all $t \in \mathbb{R}$ we have*

$$N_\rho(\tilde{\phi}_t g) = N_\rho(g) + \left(\int_0^t f_\rho(\tilde{\phi}_s(g)) ds \right) \nu(g).$$

We call N_ρ a *neutralised section.* Using the existence of a neutralised section Goldman–Labourie proved the following theorem in [17]:

Theorem 3.2.2. *[Goldman–Labourie] Let $\rho : \Gamma \to \mathsf{G}$ be a homomorphism giving rise to a Margulis Space Time such that $\mathtt{L}(\rho(\Gamma))$ contains no parabolic elements. Also let $\mathsf{U}_{\mathrm{rec}}\Sigma_{\mathtt{L}_\rho}$ and $\mathsf{U}_{\mathrm{rec}}\mathsf{M}_\rho$ be defined as above. Now if N_ρ is a neutralised section, then there exists an injective map $\hat{\mathtt{N}}_\rho$ such that the following diagram commutes,*

$$\begin{array}{ccc} \mathsf{U}^\rho_{\mathrm{rec}}\mathbb{H} & \xrightarrow{\mathtt{N}_\rho} & \mathsf{U}\mathbb{A} \\ \pi \downarrow & & \downarrow \pi \\ \mathsf{U}_{\mathrm{rec}}\Sigma_{\mathtt{L}_\rho} & \xrightarrow{\hat{\mathtt{N}}_\rho} & \mathsf{UM} \end{array}$$

where $\mathtt{N}_\rho := (N_\rho, \nu)$. Moreover, $\hat{\mathtt{N}}_\rho$ is an orbit equivalent Hölder homeomorphism onto $\mathsf{U}_{\mathrm{rec}}\mathsf{M}_\rho$.

3.3 The Representation Variety

Let Γ be a free group with n generators and $\mathsf{G} = \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$. Also let

$$\begin{aligned}\rho : \Gamma &\longrightarrow \mathsf{G}\\ \gamma &\longmapsto (\mathtt{L}_\rho(\gamma), \mathtt{u}_\rho(\gamma))\end{aligned}$$

be an injective homomorphism of Γ where $\mathtt{L}_\rho(\gamma) := \mathtt{L}(\rho(\gamma))$ and $\mathtt{u}_\rho(\gamma) := \mathtt{u}(\rho(\gamma))$ for all γ in Γ. We call $\mathtt{L}_\rho$ the *linear part* of ρ and $\mathtt{u}_\rho$ the *translation part* of ρ. If ρ is an injective homomorphism of Γ into G then $\mathtt{L}_\rho$ is an injective homomorphism of Γ into $\mathsf{SO}^0(2,1))$ and $\mathtt{u}_\rho$ satisfies the cocycle identity, that is,

$$\mathtt{u}_\rho(\gamma_1.\gamma_2) = \mathtt{L}_\rho(\gamma_1)\mathtt{u}_\rho(\gamma_2) + \mathtt{u}_\rho(\gamma_1).$$

We denote the space of all injective homomorphisms from a free group Γ into a Lie group G by $\mathsf{Hom}(\Gamma, G)$ and the space of cocycles by $\mathsf{Z}^1(\mathtt{L}_\rho(\Gamma), \mathbb{R}^3)$. We denote the space of all homomorphisms ρ in $\mathsf{Hom}(\Gamma, \mathsf{G})$ such that $\rho(\Gamma)$ acts properly on $\mathbb{A}$ and $\mathtt{L}_\rho(\Gamma)$ is discrete containing no parabolic elements by $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$. We note that any homomorphism ρ in $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ gives rise to a Margulis Space Time

$$\mathsf{M}_\rho := \rho(\Gamma)\backslash\mathbb{A}.$$

Let $\mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ denote the space of all ϱ in $\mathsf{Hom}(\Gamma, \mathsf{SO}^0(2,1))$ such that $\varrho(\Gamma)$ is Schottky. We note that $\mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ is an analytic manifold and for any ϱ in $\mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ the tangent space $\mathsf{T}_\varrho\mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ of $\mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ at the point ϱ can be identified with $\mathsf{Z}^1(\varrho(\Gamma), \mathbb{R}^3)$. Also note that

$$\begin{aligned}\mathtt{L} : \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G}) &\longrightarrow \mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))\\ \rho &\longmapsto \mathtt{L}_\rho\end{aligned} \tag{3.3.1}$$

is a bundle over $\mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ with projection map given by $\mathtt{L}$. We note that $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ can be identified with a sub-bundle of the tangent bundle $\mathsf{THom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ of $\mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$.

Lemma 3.3.1. *The space* $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ *is an analytic manifold.*

Proof. We know that the space $\mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ is an analytic manifold. Hence the tangent bundle $\mathsf{THom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ is also an analytic manifold. Now from [18] we get that the set of all ρ in $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ with fixed linear part ϱ is an open convex cone in $\mathsf{T}_\varrho\mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$. Therefore we conclude that $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ is an analytic manifold. □

3.4 The Margulis Invariant

Let $\rho : \Gamma \to \mathsf{G}$ be a homomorphism such that the action of $\mathtt{L}_\rho(\Gamma)$ on $\mathbb{H}$ is Schottky. We define the *Margulis Invariant* of an element γ in Γ for a given homomorphism ρ as follows

$$\alpha_\rho(\gamma) := \left\langle \mathtt{u}_\rho(\gamma) \mid \nu_\rho\left(\gamma^-, \gamma^+\right)\right\rangle. \tag{3.4.1}$$

where $\mathtt{u}_\rho(\gamma) := \mathtt{u}(\rho(\gamma))$ and $\nu_\rho\left(\gamma^-, \gamma^+\right) := \nu\left((\mathtt{L}_\rho(\gamma))^-, (\mathtt{L}_\rho(\gamma))^+\right)$.

In [27] and [28] Margulis showed the follwing result,

Lemma 3.4.1. *[Opposite sign lemma] If $\rho : \Gamma \to \mathsf{G}$ is a homomorphism giving rise to a Margulis Space Time, then*

1. *either $\alpha_\rho(\gamma) > 0$ for all $\gamma \in \Gamma$,*
2. *or $\alpha_\rho(\gamma) < 0$ for all $\gamma \in \Gamma$.*

In [18] Goldman–Labourie–Margulis generalised the previous result and proved the following:

Theorem 3.4.2. *[Goldman–Labourie–Margulis] Let $(\varrho_0, u) : \Gamma \to \mathsf{G}$ be a homomorphism such that the action of $\varrho_0(\Gamma)$ on $\mathbb{H}$ is Schottky. Also let $\mathcal{C}_B(\Sigma_{\varrho_0})$ be the space of ϕ-invariant Borel probability measures on $\mathsf{U}\Sigma_{\varrho_0}$ and $\mathcal{C}_{per}(\Sigma_{\varrho_0}) \subset \mathcal{C}_B(\Sigma_{\varrho_0})$ be the subspace consisting of measures supported on periodic orbits. Then the following holds:*

1. *The map*

$$\begin{aligned} \mathcal{C}_{per}(\Sigma_{\varrho_0}) &\longrightarrow \mathbb{R} \\ \mu_\gamma &\longmapsto \frac{\alpha_{(\varrho_0,u)}(\gamma)}{\ell_{\varrho_0}(\gamma)}, \end{aligned}$$

where $\ell_{\varrho_0}(\gamma)$ is the length of the corresponding closed geodesic of Σ_{ϱ_0}, extends to a continuous map

$$\begin{aligned} \mathcal{C}_B(\Sigma_{\varrho_0}) &\longrightarrow \mathbb{R} \\ \mu &\longmapsto \Upsilon_{(\varrho,u)}(\mu). \end{aligned}$$

2. *Moreover, the representation (ϱ_0, u) acts properly on $\mathbb{A}$ if and only if $\Upsilon_{(\varrho,u)}(\mu) \neq 0$ for all $\mu \in \mathcal{C}_B(\Sigma_{\varrho_0})$.*

We note that the generalization of the normalized Margulis invariant as stated above was given by Labourie in [24].

Moreover, in [19] (see also [16]) Goldman–Margulis showed:

Theorem 3.4.3. *[Goldman–Margulis] Let $\{\varrho_t\} \subset \mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ be a smooth path. Then for all $\gamma \in \Gamma$ we have*

$$\left.\frac{d}{dt}\right|_{t=0} \ell_{\varrho_t}(\gamma) = \alpha_{(\varrho_0,\dot\varrho_0)}(\gamma)$$

where $\ell_{\varrho_t}(\gamma)$ is the length of the closed geodesic of Σ_{ϱ_t} corresponding to $\varrho_t(\gamma) \in \varrho_t(\Gamma)$ and $\dot\varrho_0 := \left.\frac{d}{dt}\right|_{t=0} \varrho_t$.

Chapter 4

Metric Anosov Property

The definitions in this chapter, which can also be found in subsection 3.2 of [8], has been included here for the sake of completeness.

Definition 4.0.4. *Let $(\mathcal{X}, d)$ be a metric space. A lamination $\mathcal{L}$ of $\mathcal{X}$ is an equivalence relation on $\mathcal{X}$ such that for all x in $\mathcal{X}$ there exist an open neighborhood $\mathcal{U}_x$ of x in $\mathcal{X}$, two topological spaces $\mathcal{U}_1$ and $\mathcal{U}_2$ and a homeomorphism $\mathfrak{f}_x$ from $\mathcal{U}_1 \times \mathcal{U}_2$ onto $\mathcal{U}_x$ satisfying the following properties,*

1. *for all w, z in $\mathcal{U}_x \cap \mathcal{U}_y$ we have $p_2\left(\mathfrak{f}_x^{-1}(w)\right) = p_2\left(\mathfrak{f}_x^{-1}(z)\right)$ if and only if $p_2\left(\mathfrak{f}_y^{-1}(w)\right) = p_2\left(\mathfrak{f}_y^{-1}(z)\right)$ where p_2 is the projection from $\mathcal{U}_1 \times \mathcal{U}_2$ onto $\mathcal{U}_2$,*
2. *for all w, z in $\mathcal{X}$ we have $w\mathcal{L}z$ if and only if there exists a finite sequence of points $w_1, w_2, .., w_n$ in $\mathcal{X}$ with $w_1 = w$ and $w_n = z$, such that w_{i+1} is in $\mathcal{U}_{w_i}$, where $\mathcal{U}_{w_i}$ is a neighborhood of w_i and $p_2\left(\mathfrak{f}_{w_i}^{-1}(w_i)\right) = p_2\left(\mathfrak{f}_{w_i}^{-1}(w_{i+1})\right)$ for all i in $\{1, 2, .., n-1\}$.*

The homeomorphism $\mathfrak{f}_x$ is called a *chart* and the equivalence classes are called the *leaves.*
A *plaque open set* in the chart corresponding to $\mathfrak{f}_x$ is a set of the form $\mathfrak{f}_x(\mathcal{V}_1 \times \{x_2\})$ where $x = \mathfrak{f}_x(x_1, x_2)$ and $\mathcal{V}_1$ is an open set in $\mathcal{U}_1$. The *plaque topology* on $\mathcal{L}_x$ is the topology generated by the plaque open sets. A plaque neighborhood of x is a neighborhood for the plaque topology on $\mathcal{L}_x$.

Definition 4.0.5. *A local product structure on $\mathcal{X}$ is a pair of two laminations $\mathcal{L}_1$, $\mathcal{L}_2$ satisfying the following property: for all x in $\mathcal{X}$ there exist two plaque neighborhoods $\mathcal{U}_1$, $\mathcal{U}_2$ of x, respectively in $\mathcal{L}_1$, $\mathcal{L}_2$ and a homeomorphism $\mathfrak{f}_x$ from $\mathcal{U}_1 \times \mathcal{U}_2$ onto a neighborhood $\mathcal{W}_x$ of x, such that $\mathfrak{f}_x$ defines a chart for both the laminations $\mathcal{L}_1$ and $\mathcal{L}_2$.*

Now let us assume that ψ_t be a flow on $\mathcal{X}$. A lamination $\mathcal{L}$ invariant under the flow ψ_t is called *transverse* to the flow, if for all x in $\mathcal{X}$, there exists a plaque neighborhood $\mathcal{U}_x$ of x in $\mathcal{L}_x$, a topological space $\mathcal{V}$, a positive ϵ and a homeomorphism $\mathfrak{f}_x$ from $\mathcal{U}_x \times \mathcal{V} \times (-\epsilon, \epsilon)$ onto an open neighborhood $\mathcal{W}_x$ of x in $\mathcal{X}$ satisfying the following condition:

$$\psi_t(\mathfrak{f}_x(u, v, s)) = \mathfrak{f}_x(u, v, s+t)$$

for u in $\mathcal{U}_x$, v in $\mathcal{V}$ and for s, t in the interval $(-\epsilon, \epsilon)$. Let $\mathcal{L}^{\cdot}$ be a lamination which is transverse to the flow ψ_t. We define a new lamination $\mathcal{L}^{\cdot,0}$, called the *central lamination*, starting from $\mathcal{L}^{\cdot}$ as follows, we say y, z in $\mathcal{X}$ belongs to the same equivalence class of $\mathcal{L}^{\cdot,0}$ if for some real number t, $\psi_t y$ and z belongs to the same equivalence class of $\mathcal{L}^{\cdot}$.

Definition 4.0.6. *A lamination $\mathcal{L}$ invariant under a flow ψ_t is said to contract under the flow if and only if there exists a positive real number T_0 such that for all x in $\mathcal{X}$, the following holds: there exists a chart $\mathfrak{f}_x$ of an open neighbourhood W_x of x, and for any two points y, z in W_x with y, z being in the same equivalence class of $\mathcal{L}$, we have,*

$$d(\psi_t y, \psi_t z) < \frac{1}{2} d(y, z)$$

for all $t > T_0$.

Remark 4.0.7. *We note that a lamination 'contracts under a flow' if and only if the lamination contracts exponentially under the flow.*

Definition 4.0.8. *A flow ψ_t on a compact metric space is called Metric Anosov, if and only if there exist two laminations $\mathcal{L}^+$ and $\mathcal{L}^-$ of $\mathcal{X}$ such that the following conditions hold:*

1. *$(\mathcal{L}^+, \mathcal{L}^{-,0})$ defines a local product structure on $\mathcal{X}$,*
2. *$(\mathcal{L}^-, \mathcal{L}^{+,0})$ defines a local product structure on $\mathcal{X}$,*
3. *the leaves of $\mathcal{L}^+$ are contracted by the flow,*
4. *the leaves of $\mathcal{L}^-$ are contracted by the inverse flow.*

In such a case we call $\mathcal{L}^+$, $\mathcal{L}^-$, $\mathcal{L}^{+,0}$ and $\mathcal{L}^{-,0}$ respectively the *stable*, *unstable*, *central stable* and *central unstable* laminations.

Chapter 5

Metric Anosov structure on Margulis Space Time

Let M be a Margulis Space Time. In this chapter, first we define a distance function d on $\mathsf{U}_{\text{rec}}\mathsf{M}$ such that $(\mathsf{U}_{\text{rec}}\mathsf{M}, d)$ is a metric space. Next, we define two laminations $\mathcal{L}^{\pm}$ on the metric space $(\mathsf{U}_{\text{rec}}\mathsf{M}, d)$ which are invariant under the flow Φ_t on $\mathsf{U}_{\text{rec}}\mathsf{M}$. Finally, we show that the lamination $\mathcal{L}^{+}$ is a stable lamination and the lamination $\mathcal{L}^{-}$ is an unstable lamination for the flow Φ_t on $(\mathsf{U}_{\text{rec}}\mathsf{M}, d)$. We note that the method used in this thesis to construct the distance function d and to prove contraction properties of the lamination is inspired by [8].

5.1 Metric space structure

The restriction of any euclidean metric on $\mathbb{A}\times\mathbb{V}$ to the subspace $\mathsf{U}_{\text{rec}}\mathbb{A}$, defines a distance on $\mathsf{U}_{\text{rec}}\mathbb{A}$. We call this distance the *euclidean distance* on $\mathsf{U}_{\text{rec}}\mathbb{A}$. In this section we will define a distance on the space $\mathsf{U}_{\text{rec}}\mathbb{A}$ such that the distance is locally bilipschitz equivalent to any euclidean distance on $\mathsf{U}_{\text{rec}}\mathbb{A}$ and also is Γ-invariant, so as to get a distance on the quotient space $\mathsf{U}_{\text{rec}}\mathsf{M}$.

We note that any two euclidean metric on $\mathbb{A}\times\mathbb{V}$ are bilipschitz equivalent with each other and hence any two euclidean distances on $\mathsf{U}_{\text{rec}}\mathbb{A}$ are also bilipschitz equivalent with each other. Fix an euclidean distance d on $\mathsf{U}_{\text{rec}}\mathbb{A}$. The action of Γ on the space $\mathbb{A}\times\mathbb{V}$ gives rise to a collection of distances related to d defined as follows: for any γ in Γ define,

$$\begin{aligned} d_\gamma : \mathsf{U}_{\text{rec}}\mathbb{A}\times\mathsf{U}_{\text{rec}}\mathbb{A} &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto d\left(\gamma^{-1}x, \gamma^{-1}y\right) \end{aligned} \tag{5.1.1}$$

Since each element of Γ acts as a bilipschitz automorphism with respect to any euclidean distance, any two distances in the family $\{d_\gamma\}_{\gamma\in\Gamma}$ are bilipschitz equivalent with each other.

Compactness of $\mathsf{U}_{\text{rec}}\Sigma$ implies that $\mathsf{U}_{\text{rec}}\mathsf{M}$ is compact and hence we can choose a precompact fundamental domain D of $\mathsf{U}_{\text{rec}}\mathsf{M}$ inside $\mathsf{U}_{\text{rec}}\mathbb{A}$ with an open interior. We can also choose a suitable precompact open set U which contains the closure of D. We note that properness of the action of Γ on $\mathsf{U}_{\text{rec}}\mathbb{A}$ implies that the cover of $\mathsf{U}_{\text{rec}}\mathbb{A}$ by the open sets $\{\gamma.U\}_{\gamma\in\Gamma}$, is locally finite.

A path joining two points x and y in $\mathsf{U}_{\text{rec}}\mathbb{A}$ is a pair of tuples,

$$\mathcal{P} = ((z_0, z_1, .., z_n), (\gamma_1, \gamma_2, .., \gamma_n))$$

where $z_i \in \mathsf{U}_{\text{rec}}\mathbb{A}$ and $\gamma_i \in \Gamma$ such that the following two conditions hold,

1. $x = z_0 \in \gamma_1.U$ and $y = z_n \in \gamma_n.U$,
2. for all $n > i > 0, z_i \in \gamma_i.U \cap \gamma_{i+1}.U$.

Definition 5.1.1. *The length of a path is defined by,*

$$l(\mathcal{P}) := \sum_{i=0}^{n-1} d_{\gamma_{i+1}}(z_i, z_{i+1})$$

Definition 5.1.2. *We then define,*

$$\tilde{d}(x, y) := \mathit{inf}\,\{l(\mathcal{P}) \mid \mathcal{P} \textit{ joins } x \textit{ and } y\}$$

Lemma 5.1.3. *$\tilde{d}$ is a Γ-invariant pseudo-metric.*

Proof. If $\mathcal{P} = ((z_0, z_1, .., z_n), (\gamma_1, \gamma_2, .., \gamma_n))$ is a path joining γx and γy, then the path,

$$\gamma^{-1}.\mathcal{P} := \left(\left(\gamma^{-1}z_0, \gamma^{-1}z_1, .., \gamma^{-1}z_n\right), \left(\gamma^{-1}\gamma_1, \gamma^{-1}\gamma_2, .., \gamma^{-1}\gamma_n\right)\right)$$

is a path joining x and y. Moreover,

$$\begin{aligned}
l(\mathcal{P}) &= \sum_{i=0}^{n-1} d_{\gamma_{i+1}}(z_i, z_{i+1}) = \sum_{i=0}^{n-1} d\left(\gamma_{i+1}^{-1}z_i, \gamma_{i+1}^{-1}z_{i+1}\right) \\
&= \sum_{i=0}^{n-1} d\left(\left(\gamma^{-1}\gamma_{i+1}\right)^{-1}\gamma^{-1}z_i, \left(\gamma^{-1}\gamma_{i+1}\right)^{-1}\gamma^{-1}z_{i+1}\right) \\
&= \sum_{i=0}^{n-1} d_{\gamma^{-1}\gamma_{i+1}}\left(\gamma^{-1}z_i, \gamma^{-1}z_{i+1}\right) \\
&= l\left(\gamma^{-1}.\mathcal{P}\right).
\end{aligned}$$

Hence, using the definition of $\tilde{d}$ we get $\tilde{d}(\gamma x, \gamma y)$ is equal to $\tilde{d}(x, y)$.
We also notice that $l(\mathcal{P})$ is a sum of distances. So $l(\mathcal{P})$ is non-negative and hence $\tilde{d}$ is non-negative. ∎

It remains to show that $\tilde{d}$ is a metric and $\tilde{d}$ is locally bilipschitz equivalent to any euclidean distance. As all euclidean distances are bilipschitz equivalent with each other, it suffices to show that $\tilde{d}$ is locally bilipschitz equivalent with d.

Lemma 5.1.4. *$\tilde{d}$ is a metric and $\tilde{d}$ is locally bilipschitz equivalent to d.*

Proof. Let z be a point in $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$. There exists a neighbourhood V of z in $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ such that

$$A := \{\gamma \mid \gamma.U \cap V \neq \emptyset\}$$

is a finite set. We fix V and choose a positive real number α so that

$$\bigcup_{\gamma\in A} \{x \mid d_\gamma(z,x) \leqslant \alpha\} \subset V.$$

We have seen that any two distances in the family $\{d_\gamma\}_{\gamma\in\Gamma}$ are bilipschitz equivalent with each other. Hence A being a subset of Γ, any two distances in A are bilipschitz equivalent with each other. Now finiteness of A implies that we can choose a constant K such that for all β_1, β_2 in A we have that d_{β_1} and d_{β_2} are K-bilipschitz equivalent with each other. We set,

$$W := \bigcap_{\gamma\in A} \left\{x \mid d_\gamma(z,x) \leqslant \frac{\alpha}{10K}\right\}.$$

We note that W is a subset of V because K is bigger than 1.

By construction, if x, y is in W then for all γ in A we have,

$$d_\gamma(x,y) \leqslant d_\gamma(x,z) + d_\gamma(z,y) \leqslant \frac{\alpha}{5K}. \tag{5.1.2}$$

Now let x be any point in W, y be any general point and

$$\mathcal{P} = ((z_0, z_1, .., z_n), (\gamma_1, \gamma_2, .., \gamma_n))$$

be a path joining x and y.

We notice that $x = z_0$ is in $\gamma_1 U$. On the other hand x is also an element of W, which is a subset of V. Therefore,

$$\gamma_1 U \cap V \neq \emptyset$$

Hence γ_1 is in A. If there exists k such that γ_k is not in A then we choose j to be the smallest k such that γ_k is not in A.

$$l(\mathcal{P}) = \sum_{i=0}^{n-1} d_{\gamma_{i+1}}(z_i, z_{i+1}) \geqslant \sum_{i=0}^{j-1} d_{\gamma_{i+1}}(z_i, z_{i+1}). \tag{5.1.3}$$

Now using the fact that $d_{\gamma_{j-1}}$ is K-bilipschitz equivalent with d_{γ_i} for any γ_i in A we get,

$$\sum_{i=0}^{j-1} d_{\gamma_{i+1}}(z_i, z_{i+1}) \geqslant \frac{1}{K}\sum_{i=0}^{j-1} d_{\gamma_{j-1}}(z_i, z_{i+1}). \tag{5.1.4}$$

Now from the triangle inequality it follows that

$$\begin{aligned}\frac{1}{K}\sum_{i=0}^{j-1} d_{\gamma_{j-1}}(z_i, z_{i+1}) &\geqslant \frac{1}{K} d_{\gamma_{j-1}}(z_0, z_j) \\ &\geqslant \frac{1}{K}(d_{\gamma_{j-1}}(z, z_j) - d_{\gamma_{j-1}}(z, z_0)).\end{aligned} \tag{5.1.5}$$

The point $z_0 = x$, belongs to W and γ_{j-1} belongs to A. Therefore, by the definition of W we get that

$$d_{\gamma_{j-1}}(z, z_0) \leqslant \frac{\alpha}{10K}. \tag{5.1.6}$$

We also know that γ_j is not in A. Hence $\gamma_j.U$ does not intersect with V. The point z_j by definition belongs to $\gamma_j.U$ and so z_j is not in V. Therefore by the choice of α it follows that

$$d_{\gamma_{j-1}}(z, z_j) > \alpha. \tag{5.1.7}$$

Using the inequalities 5.1.3 and 5.1.6 we get that

$$\frac{1}{K}\left(d_{\gamma_{j-1}}(z, z_j) - d_{\gamma_{j-1}}(z, z_0)\right) > \frac{1}{K}\left(\alpha - \frac{\alpha}{10K}\right). \tag{5.1.8}$$

Now as K is bigger than 1 we have,

$$\frac{1}{K}\left(\alpha - \frac{\alpha}{10K}\right) \geqslant \frac{1}{K}\left(\alpha - \frac{\alpha}{10}\right) > \frac{\alpha}{5K}. \tag{5.1.9}$$

Finally, using the inequalities from 5.1.3 to 5.1.9 we get that if there exists k such that γ_k is not in A then,

$$l(\mathcal{P}) > \frac{\alpha}{5K}. \tag{5.1.10}$$

On the other hand, if for all k we have γ_k in A, then for all $\gamma \in A$ we have,

$$l(\mathcal{P}) = \sum_{i=0}^{n-1} d_{\gamma_{i+1}}(z_i, z_{i+1}) \geqslant \frac{1}{K}\sum_{i=0}^{n-1} d_{\gamma}(z_i, z_{i+1}). \tag{5.1.11}$$

And using triangle inequality it follows that

$$\frac{1}{K}\sum_{i=0}^{n-1} d_{\gamma}(z_i, z_{i+1}) \geqslant \frac{1}{K} d_{\gamma}(x, y). \tag{5.1.12}$$

Therefore, in the case when for all k, γ_k is in A, we have for all γ in A,

$$l(\mathcal{P}) \geqslant \frac{1}{K} d_{\gamma}(x, y). \tag{5.1.13}$$

Combining the inequalities 5.1.10 and 5.1.13 and using the definition of $\tilde{d}$ we have that for any point x in W, any general point y and for all γ in A,

$$\tilde{d}(x, y) \geqslant \frac{1}{K}\inf\left(\frac{\alpha}{5}, d_{\gamma}(x, y)\right). \tag{5.1.14}$$

Therefore for any point y distinct from z we have,

$$\tilde{d}(z, y) > 0. \tag{5.1.15}$$

The above is true for any arbitrary choice of z and hence it follows that $\tilde{d}$ is a metric.

Moreover, if x, y are points in W and γ is in A then from the inequality 5.1.2 we get,

$$d_\gamma(x, y) \leqslant \frac{\alpha}{5K} \leqslant \frac{\alpha}{5}$$

and hence for all x, y in W and γ in A,

$$\inf\left(\frac{\alpha}{5}, d_\gamma(x, y)\right) = d_\gamma(x, y). \tag{5.1.16}$$

Therefore, from the inequalities 5.1.14 and 5.1.16 it follows that for x, y in W and for any γ in A,

$$\tilde{d}(x, y) \geqslant \frac{1}{K} d_\gamma(x, y). \tag{5.1.17}$$

We know that there exists γ_a such that the point z is inside the open set $\gamma_a.U$. We note that the above defined γ_a is also an element of A. Finally, we set W_a to be the intersection of of the set W with the set $\gamma_a.U$. Let x, y be any two points in W_a. We choose the path $\mathcal{P}_0 = ((x, y), (\gamma_a, \gamma_a))$ and get that

$$\tilde{d}(x, y) = \inf\left\{l(\mathcal{P}) \mid \mathcal{P} \text{ joins } x \text{ and } y\right\} \leqslant l(\mathcal{P}_0) = d_{\gamma_a}(x, y).$$

Hence, $\tilde{d}$ is bilipschitz equivalent to d_{γ_a} on W_a and the distance d is bilipschitz equivalent to d_{γ_a}. Therefore, d is bilipschitz to $\tilde{d}$ on W_a. Since z was arbitrarily chosen it follows that d is locally bilipschitz equivalent to $\tilde{d}$.

□

5.2 The lamination and its lift

In this section, we explicitly describe two laminations of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ for the flow Φ_t on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ and show that the laminations are equivariant under the action of the flow and the action of Γ. We will also define the notion of a leaf lift.

Let Z be a point in $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$. We know from the theorem 3.2.2 that for all $Z \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ there exists an unique $g \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ such that $Z = \mathtt{N}(g)$.

Definition 5.2.1. *The positive and central positive partition of* $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ *are respectively given by,*

$$\mathcal{L}^{+}_{\mathtt{N}(g)} := \tilde{\mathcal{L}}^{+}_{\mathtt{N}(g)} \cap \mathsf{U}_{\mathrm{rec}}\mathbb{A}$$
$$\mathcal{L}^{+,0}_{\mathtt{N}(g)} := \tilde{\mathcal{L}}^{+,0}_{\mathtt{N}(g)} \cap \mathsf{U}_{\mathrm{rec}}\mathbb{A}$$

where

$$\tilde{\mathcal{L}}^{+}_{\mathtt{N}(g)} := \{(N(g) + s_1\nu^+(g), \nu(g) + s_2\nu^+(g)) \mid s_1, s_2 \in \mathbb{R}\},$$
$$\tilde{\mathcal{L}}^{+,0}_{\mathtt{N}(g)} := \{(N(g) + s_1\nu^+(g) + t\nu(g), \nu(g) + s_2\nu^+(g)) \mid t, s_1, s_2 \in \mathbb{R}\}.$$

Definition 5.2.2. *The negative and central negative partition of* $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ *are respectively given by,*

$$\mathcal{L}^{-}_{\mathtt{N}(g)} := \tilde{\mathcal{L}}^{-}_{\mathtt{N}(g)} \cap \mathsf{U}_{\mathrm{rec}}\mathbb{A}$$
$$\mathcal{L}^{-,0}_{\mathtt{N}(g)} := \tilde{\mathcal{L}}^{-,0}_{\mathtt{N}(g)} \cap \mathsf{U}_{\mathrm{rec}}\mathbb{A}$$

where

$$\tilde{\mathcal{L}}^{-}_{\mathtt{N}(g)} := \{(N(g) + s_1\nu^-(g), \nu(g) + s_2\nu^-(g)) \mid s_1, s_2 \in \mathbb{R}\},$$
$$\tilde{\mathcal{L}}^{-,0}_{\mathtt{N}(g)} := \{(N(g) + s_1\nu^-(g) + t\nu(g), \nu(g) + s_2\nu^-(g)) \mid t, s_1, s_2 \in \mathbb{R}\}.$$

Lemma 5.2.3. *Let* g, h *be two points in* $\mathsf{U}\mathbb{H}$ *then*

$$h \text{ is in } \bigcup_{t\in\mathbb{R}} \tilde{\mathcal{H}}^{+}_{\tilde{\phi}_t g} \text{ if and only if } \nu(h) = \nu(g) + \frac{\langle \nu(h), \nu^-(g)\rangle}{\langle \nu^+(g), \nu^-(g)\rangle}\nu^+(g).$$

Proof. Let h be a point of $\bigcup_{t\in\mathbb{R}} \tilde{\mathcal{H}}^{+}_{\tilde{\phi}_t g}$. Hence there exist real numbers t_1, t_2 such that $h = \nu(ga(t_1)u^+(t_2))$. Therefore, we have

$$\begin{aligned}
\nu(h) = \nu(ga(t_1)u^+(t_2)) &= ga(t_1)u^+(t_2)\begin{pmatrix}1\\0\\0\end{pmatrix} = ga(t_1)\begin{pmatrix}1\\2t_2\\2t_2\end{pmatrix}\\
&= ga(t_1)\left(\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\2t_2\\2t_2\end{pmatrix}\right) = \nu(g) + 2t_2.ga(t_1)\begin{pmatrix}0\\1\\1\end{pmatrix}\\
&= \nu(g) + 2t_2(\cosh t_1 + \sinh t_1).g\begin{pmatrix}0\\1\\1\end{pmatrix}\\
&= \nu(g) + 2\sqrt{2}\ t_2(\cosh t_1 + \sinh t_1).\nu^+(g).
\end{aligned}$$

Now we notice that

$$\begin{aligned}
\langle \nu(h), \nu^-(g)\rangle &= \langle \nu(g) + 2\sqrt{2}\ t_2(\cosh t_1 + \sinh t_1).\nu^+(g), \nu^-(g)\rangle\\
&= 2\sqrt{2}\ t_2(\cosh t_1 + \sinh t_1).\langle \nu^+(g), \nu^-(g)\rangle.
\end{aligned}$$

Combining the above two calculations we get

$$\nu(h) = \nu(g) + \frac{\langle \nu(h), \nu^-(g)\rangle}{\langle \nu^+(g), \nu^-(g)\rangle}\nu^+(g).$$

Now let g, h be two points in $\mathsf{U}\mathbb{H}$ satisfying,

$$\nu(h) = \nu(g) + a_1\nu^+(g)$$

for some real number a_1. Using the definition of ν and ν^+ we observe that the above equation is equvalent to the following equation,

$$\left(g.u^+\left(\frac{a_1}{2\sqrt{2}}\right)\right)^{-1}.h\begin{pmatrix}1\\0\\0\end{pmatrix}=\left(u^+\left(\frac{a_1}{2\sqrt{2}}\right)\right)^{-1}\begin{pmatrix}1\\a_1/\sqrt{2}\\a_1/\sqrt{2}\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

We know that the only elements of $\mathsf{SO}^0(2,1)$ fixing the vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$ are of the form $a(t)$ for some real number t. Hence there exist a real number t_1 such that

$$\left(g.u^+\left(\frac{a_1}{2\sqrt{2}}\right)\right)^{-1}.h=a(t_1).$$

Therefore,

$$h=g.u^+\left(\frac{a_1}{2\sqrt{2}}\right).a(t_1)=g.a(t_1).u^+\left(\frac{a_1\exp(-t_1)}{2\sqrt{2}}\right)$$

and the result follows. □

Corollary 5.2.4. *Let g,h be two points in $\mathsf{U}\mathbb{H}$ and h is in $\bigcup_{t\in\mathbb{R}}\tilde{\mathcal{H}}^+_{\tilde{\phi}_t g}$ then*

$$\frac{\langle\nu(g),\nu^-(h)\rangle}{\langle\nu^+(h),\nu^-(h)\rangle}\nu^+(h)=-\frac{\langle\nu(h),\nu^-(g)\rangle}{\langle\nu^+(g),\nu^-(g)\rangle}\nu^+(g).$$

Proof. We know that if h is in $\bigcup_{t\in\mathbb{R}}\tilde{\mathcal{H}}^+_{\tilde{\phi}_t g}$ then g is in $\bigcup_{t\in\mathbb{R}}\tilde{\mathcal{H}}^+_{\tilde{\phi}_t h}$. Therefore using lemma 5.2.3 we get

$$\nu(h)=\nu(g)+\frac{\langle\nu(h),\nu^-(g)\rangle}{\langle\nu^+(g),\nu^-(g)\rangle}\nu^+(g)$$

and

$$\nu(g)=\nu(h)+\frac{\langle\nu(g),\nu^-(h)\rangle}{\langle\nu^+(h),\nu^-(h)\rangle}\nu^+(h).$$

Hence

$$\frac{\langle\nu(g),\nu^-(h)\rangle}{\langle\nu^+(h),\nu^-(h)\rangle}\nu^+(h)=-\frac{\langle\nu(h),\nu^-(g)\rangle}{\langle\nu^+(g),\nu^-(g)\rangle}\nu^+(g).$$

□

Definition 5.2.5. *For all g in $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$ we define,*

$$\mathcal{H}^\pm_g:=\tilde{\mathcal{H}}^\pm_g\cap\mathsf{U}_{\mathrm{rec}}\mathbb{H}.$$

Proposition 5.2.6. *The following equations are true for all g in $\mathsf{U}_{\text{rec}}\mathbb{H}$,*

1. $\mathcal{L}^{+,0}_{\mathtt{N}(g)} = \left\{ \mathtt{N}(h) \mid h \in \bigcup_{t\in\mathbb{R}} \mathcal{H}^{+}_{\tilde{\phi}_t g} \right\},$
2. $\mathcal{L}^{-,0}_{\mathtt{N}(g)} = \left\{ \mathtt{N}(h) \mid h \in \bigcup_{t\in\mathbb{R}} \mathcal{H}^{-}_{\tilde{\phi}_t g} \right\}.$

Proof. We start with defining a function,

$$\begin{aligned} F : \mathsf{U}_{\text{rec}}\mathbb{H} \times \mathsf{U}_{\text{rec}}\mathbb{H} &\to \mathbb{R} \\ (g,h) &\mapsto \det[(N(g) - N(h)), \nu(g), \nu(h)]. \end{aligned} \tag{5.2.1}$$

Using equation 2.3.3 and theorem 3.2.1 we get that

$$F(\tilde{\phi}_t g, \tilde{\phi}_t h) = F(g,h) \tag{5.2.2}$$

for all $t \in \mathbb{R}$. Again using equation 2.3.4 and theorem 3.2.1 we get that the neutralised section and the neutral section are equivariant under the action of Γ. Hence for all γ in Γ we have,

$$\begin{aligned} F(\gamma g, \gamma h) &= \det[(N(\gamma g) - N(\gamma h)), \nu(\gamma g), \nu(\gamma h)] \\ &= \det[\gamma(N(g) - N(h)), \gamma\nu(g), \gamma\nu(h))] \\ &= \det[\gamma] \det[(N(g) - N(h)), \nu(g), \nu(h)] \\ &= \det[(N(g) - N(h)), \nu(g), \nu(h)] \\ &= F(g,h). \end{aligned} \tag{5.2.3}$$

Now for a fixed real number c_0 we consider the space,

$$\mathfrak{K} := \{(g_1, g_2) \mid d_{\mathsf{U}\mathbb{H}}(g_1, g_2) \leqslant c_0\} \subset \mathsf{U}_{\text{rec}}\mathbb{H} \times \mathsf{U}_{\text{rec}}\mathbb{H}.$$

Compactness of $\mathsf{U}_{\text{rec}}\Sigma$ implies that $\mathfrak{K}_\Gamma$, the projection of $\mathfrak{K}$ in $\Gamma\backslash(\mathsf{U}_{\text{rec}}\mathbb{H}\times\mathsf{U}_{\text{rec}}\mathbb{H})$, is compact. Now continuity of F implies that F is uniformly continuous on $\mathfrak{K}_\Gamma$.

Let g and h be two points in $\mathsf{U}_{\text{rec}}\mathbb{H}$ such that h is in $\mathcal{H}^+_g$. Given any such choice of g and h we can choose a sufficiently large t_0 such that $d_{\mathsf{U}\mathbb{H}}(\tilde{\phi}_{t_0} g, \tilde{\phi}_{t_0} h)$ is arbitrarily close to zero, hence we have $F(\tilde{\phi}_{t_0} g, \tilde{\phi}_{t_0} h)$ arbitrarily close to zero. Therefore by using equation 5.2.2 it follows that $F(g,h)$ is zero for all h in $\mathcal{H}^+_g$.

Now using equation 5.2.2, equation 2.3.3 and lemma 5.2.3 we have,

$$\begin{aligned} 0 = F(\tilde{\phi}_t g, \tilde{\phi}_t h) &= \det[(N(\tilde{\phi}_t g) - N(\tilde{\phi}_t h)), \nu(\tilde{\phi}_t g), \nu(\tilde{\phi}_t h)] \\ &= \det[(N(\tilde{\phi}_t g) - N(\tilde{\phi}_t h)), \nu(g), \nu(h)] \\ &= \det[(N(\tilde{\phi}_t g) - N(\tilde{\phi}_t h)), \nu(g), \nu(g) + \frac{\langle \nu(h), \nu^-(g)\rangle}{\langle \nu^+(g), \nu^-(g)\rangle}\nu^+(g)] \\ &= \frac{\langle \nu(h), \nu^-(g)\rangle}{\langle \nu^+(g), \nu^-(g)\rangle} \det[(N(\tilde{\phi}_t g) - N(\tilde{\phi}_t h)), \nu(g), \nu^+(g)]. \end{aligned}$$

Therefore for all h in $\mathcal{H}_g^+$ and for all real number t we have

$$\det[(N(\tilde{\phi}_t g) - N(\tilde{\phi}_t h)), \nu(g), \nu^+(g)] = 0.$$

Hence there exist real numbers a_1, b_1 such that

$$\begin{aligned} N(\tilde{\phi}_t h) &= N(\tilde{\phi}_t g) + a_1\nu(g) + b_1\nu^+(g) \\ &= N(g) + \left(a_1 + \int_0^t f(\tilde{\phi}_s(g))ds \right) \nu(g) + b_1\nu^+(g). \end{aligned} \tag{5.2.4}$$

Combining lemma 5.2.3 and equation 5.2.4 we get that

$$\mathcal{L}^{+,0}_{\mathtt{N}(g)} \supseteq \left\{ \mathtt{N}(h) \mid h \in \bigcup_{t\in\mathbb{R}} \mathcal{H}^+_{\tilde{\phi}_t g} \right\}$$

Now let $W \in \mathcal{L}^{+,0}_{\mathtt{N}(g)}$. By theorem 3.2.2 we know that there exist $h \in \mathsf{U}_{\text{rec}}\mathbb{H}$ such that $W = \mathtt{N}(h)$. Now the choice of W implies that there exist some real number a_2 such that

$$\nu(h) = \nu(g) + a_2\nu^+(g).$$

Using lemma 5.2.3 we get that $h \in \bigcup_{t\in\mathbb{R}} \tilde{\mathcal{H}}^+_{\tilde{\phi}_t g}$. Therefore h is in

$$\bigcup_{t\in\mathbb{R}} \mathcal{H}^+_{\tilde{\phi}_t g} = \left(\mathsf{U}_{\text{rec}}\mathbb{H} \cap \bigcup_{t\in\mathbb{R}} \tilde{\mathcal{H}}^+_{\tilde{\phi}_t g} \right)$$

and we have

$$\mathcal{L}^{+,0}_{\mathtt{N}(g)} \subseteq \left\{ \mathtt{N}(h) \mid h \in \bigcup_{t\in\mathbb{R}} \mathcal{H}^+_{\tilde{\phi}_t g} \right\}.$$

Similarly the other equality follows. $\square$

Proposition 5.2.7. *Let $\mathcal{U}_{\mathtt{N}(g)} \subset \mathsf{U}_{\text{rec}}\mathbb{A}$ be a neighborhood of a point $\mathtt{N}(g)$ in $\mathsf{U}_{\text{rec}}\mathbb{A}$. Then the following map is a local homeomorphism:*

$$\begin{aligned} \amalg_{\mathtt{N}(g)} : \mathcal{U}_{\mathtt{N}(g)} &\to (\Lambda_\infty\Gamma \times \Lambda_\infty\Gamma \setminus \Delta) \times \mathbb{R} \\ \mathtt{N}(h) &\mapsto \left(h^-, h^+, \left\langle N(h) - N(g), \nu\left(g^-, h^+\right)\right\rangle\right) \end{aligned}$$

where $h^\pm := \lim_{t\to\pm\infty} \pi(\tilde{\phi}_t h)$ and π is the projection from $\mathsf{U}\mathbb{H}$ onto $\mathbb{H}$.

Proof. Let g be a point in $\mathsf{U}_{\text{rec}}\mathbb{H}$. We note that for $g \in \mathsf{U}_{\text{rec}}\mathbb{H}$ the points $g^\pm$ lies in $\Lambda_\infty\Gamma$. We observe that $\partial\mathbb{H} \setminus \{g^+\}$ is homeomorpic to $\mathbb{R}$. Given any g, let $\mathcal{V}_{g^-}$ denote a connected bounded open neighborhood of g^- in $\partial\mathbb{H} \setminus \{g^+\}$ and $\mathcal{V}_{g^+}$ be a connected open neighborhood of g^+ in $\partial\mathbb{H} \setminus \{g^-\}$ such that $\mathcal{V}_{g^-} \cap \mathcal{V}_{g^+}$ is empty and $\mathcal{V}_{g^-} \times \mathcal{V}_{g^+}$ is a subset of $\partial\mathbb{H}\times\partial\mathbb{H}\setminus\Delta$. We define $\mathcal{U}_{g^\pm} := \mathcal{V}_{g^\pm}\cap\Lambda_\infty\Gamma$. Let $\mathcal{U}_g$ be the open subset of $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$ corresponding to the open set $\mathcal{U}_{g^-}\times\mathcal{U}_{g^+}\times\mathbb{R}$. We consider the following continuous map,

$$\begin{aligned}\mathfrak{N}_g : \mathcal{U}_g &\longrightarrow \mathbb{A}\\ h &\longmapsto N(h) - \left\langle N(h)-N(g), \nu(g^-,h^+)\right\rangle \nu(h)\end{aligned}$$

We notice that

$$\nu(g^-,h^+) = \frac{\nu^-(g)\boxtimes\nu^+(h)}{\langle\nu^-(g),\nu^+(h)\rangle}.$$

Hence for all real number t we have

$$\mathfrak{N}_g(\tilde{\phi}_t h) = \mathfrak{N}_g(h).$$

Now we define the following continuous map:

$$\begin{aligned}\Pi_g : \mathcal{U}_{g^-}\times\mathcal{U}_{g^+}\times\mathbb{R} &\longrightarrow \mathsf{U}_{\mathrm{rec}}\mathbb{A}\\ (h^-,h^+,t) &\longmapsto (\mathfrak{N}_g + t\nu, \nu)\,(h^-,h^+,t)\end{aligned}$$

and conclude by observing that

$$\begin{aligned}\amalg_{\mathtt{N}(g)}\circ\Pi_g &= \mathsf{Id},\\ \Pi_g\circ\amalg_{\mathtt{N}(g)} &= \mathsf{Id}.\end{aligned}$$

□

Proposition 5.2.8. *Let $\mathcal{L}^+$ be as defined in definition 5.2.1. Then $\mathcal{L}^+$ is a lamination of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$.*

Proof. We now show that the equivalence relation $\mathcal{L}^+$ on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ satisfy properties (1) and (2) of definition 4.0.4 for the local homeomorphism $\amalg$.

Property (1): Let g_1, g_2 be two points in $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$, h_1, h_2 be two points in the intersection $\mathcal{U}_{g_1}\cap\mathcal{U}_{g_2}$ and $p^{+,0}$ be the projection from $\mathcal{U}_{g^-}\times\mathcal{U}_{g^+}\times\mathbb{R}$ onto $\mathcal{U}_{g^+}\times\mathbb{R}$. We notice that if

$$p^{+,0}\circ\amalg_{\mathtt{N}(g_1)}(\mathtt{N}(h_1)) = p^{+,0}\circ\amalg_{\mathtt{N}(g_1)}(\mathtt{N}(h_2))$$

then $h_1^+ = h_2^+$ and

$$\left\langle N(h_1)-N(g_1), \nu(g_1^-,h_1^+)\right\rangle = \left\langle N(h_2)-N(g_1), \nu(g_1^-,h_2^+)\right\rangle.$$

Now using proposition 5.2.6, corollary 5.2.4 and the fact that $h_1^+ = h_2^+$ we get

$$\nu^+(h_2) = c\nu^+(h_1)$$

$$N(h_2) = N(h_1) + s\nu^+(h_1) + t\nu(h_1)$$

where $c, s, t\in\mathbb{R}$. Hence for $i\in\{1,2\}$

$$\nu(g_i^-,h_2^+) = \nu(g_i^-,h_1^+).$$

Finally using the fact that

$$\left\langle N(h_2)-N(h_1), \nu(g_1^-,h_1^+)\right\rangle = 0$$

and

$$\nu(g^-, h^+) = \frac{\nu^-(g) \boxtimes \nu^+(h)}{\langle \nu^-(g), \nu^+(h) \rangle}$$

we get $t = 0$. Therefore

$$\left\langle N(h_2) - N(h_1), \nu(g_2^-, h_1^+) \right\rangle = \left\langle s\nu^+(h_1), \frac{\nu^-(g_2) \boxtimes \nu^+(h_1)}{\langle \nu^-(g_2), \nu^+(h_1) \rangle} \right\rangle = 0.$$

Hence

$$\left\langle N(h_1) - N(g_2), \nu(g_2^-, h_1^+) \right\rangle = \left\langle N(h_2) - N(g_2), \nu(g_2^-, h_2^+) \right\rangle$$

and it follows that

$$p^{+,0} \circ \amalg_{\mathtt{N}(g_2)}(\mathtt{N}(h_1)) = p^{+,0} \circ \amalg_{\mathtt{N}(g_2)}(\mathtt{N}(h_2)).$$

Similarly if we have

$$p^{+,0} \circ \amalg_{\mathtt{N}(g_2)}(\mathtt{N}(h_1)) = p^{+,0} \circ \amalg_{\mathtt{N}(g_2)}(\mathtt{N}(h_2))$$

then

$$p^{+,0} \circ \amalg_{\mathtt{N}(g_1)}(\mathtt{N}(h_1)) = p^{+,0} \circ \amalg_{\mathtt{N}(g_1)}(\mathtt{N}(h_2)).$$

Property (2): Let $\{\mathtt{N}(h_i)\}_{i \in \{1,2,..,n\}}$ be a sequence of points such that for all $i \in \{1, 2, .., n-1\}$ we have

$$\mathtt{N}(h_{i+1}) \in \mathcal{U}_{\mathtt{N}(h_i)}$$

and

$$p^{+,0} \circ \amalg_{\mathtt{N}(h_i)}(\mathtt{N}(h_i)) = p^{+,0} \circ \amalg_{\mathtt{N}(h_i)}(\mathtt{N}(h_{i+1})).$$

Hence we have $h_i^+ = h_{i+1}^+$ and

$$0 = \left\langle N(h_i) - N(h_i), \nu(h_i^-, h_i^+) \right\rangle = \left\langle N(h_{i+1}) - N(h_i), \nu(h_i^-, h_{i+1}^+) \right\rangle.$$

Now using proposition 5.2.6, corollary 5.2.4 and $h_i^+ = h_{i+1}^+$ we get that

$$\nu^+(h_{i+1}) = c_i \nu^+(h_i),$$

$$N(h_{i+1}) = N(h_i) + s_i \nu^+(h_i) + t_i \nu(h_i)$$

for some real numbers c_i, s_i and t_i. Hence

$$\nu(h_i^-, h_i^+) = \nu(h_i^-, h_{i+1}^+).$$

Now using the fact that

$$\left\langle N(h_{i+1}) - N(h_i), \nu(h_i^-, h_{i+1}^+) \right\rangle = 0$$

we get $t = 0$. Hence we have

$$\mathcal{L}^{+}_{\mathtt{N}(h_i)} = \mathcal{L}^{+}_{\mathtt{N}(h_{i+1})}.$$

Therefore we conclude that

$$\mathcal{L}^{+}_{\mathtt{N}(h_1)} = \mathcal{L}^{+}_{\mathtt{N}(h_n)}.$$

Now we show the other direction. Let $h \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ such that $\mathtt{N}(h) \in \mathcal{L}^{+}_{\mathtt{N}(g)}$. Using proposition 5.2.6 we get that $h^+ = g^+$. Let $\mathcal{V}_{g^-}$ be a connected bounded open neighborhood of g^- in $\partial_\infty\mathbb{H} \setminus \{g^+\}$ containing the point h^- and let $\mathcal{V}_{g^+}$ be a connected open neighborhood of g^+ in $\partial_\infty\mathbb{H} \setminus \{g^-\}$ such that the intersection $\mathcal{V}_{g^+} \cap \mathcal{V}_{g^-}$ is empty. We denote the sets $\mathcal{V}_{g^\pm} \cap \Lambda_\infty\Gamma$ respectively by $\mathcal{U}_{g^\pm}$, the open subset of $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$ corresponding to the open set $\mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \times \mathbb{R}$ by $\mathcal{U}_g$ and the open set $\mathtt{N}(\mathcal{U}_g)$ around $\mathtt{N}(g)$ by $\mathcal{U}_{\mathtt{N}(g)}$. Now we consider the chart $\big(\mathcal{U}_{\mathtt{N}(g)}, \amalg_{\mathtt{N}(g)}\big)$ and notice that

$$p^{+,0} \circ \amalg_{\mathtt{N}(g)}(\mathtt{N}(g)) = \big(g^+, 0\big).$$

Since $\mathtt{N}(h) \in \mathcal{L}^{+}_{\mathtt{N}(g)}$, using the definition of $\mathcal{L}^{+}_{\mathtt{N}(g)}$ we get

$$\big\langle N(h) - N(g), \nu(g^-, g^+)\big\rangle = 0.$$

Now using corollary 5.2.4 and the fact that $h^+ = g^+$ we obtain

$$\nu(g^-, g^+) = \nu(g^-, h^+).$$

Hence

$$\big\langle N(h) - N(g), \nu(g^-, h^+)\big\rangle = 0$$

and we finally have

$$p^{+,0} \circ \amalg_{\mathtt{N}(g)}(\mathtt{N}(g)) = p^{+,0} \circ \amalg_{\mathtt{N}(g)}(\mathtt{N}(h)).$$

Therefore we conclude that $\mathcal{L}^+$ defines a lamination with plaque neighborhoods given by the image of the open sets $\mathcal{U}_{g^-}$ for g^- in $\Lambda_\infty\Gamma \setminus \{g^+\}$. $\square$

Proposition 5.2.9. *Let $\mathcal{L}^{-,0}$ be as defined in definition 5.2.2. Then $\mathcal{L}^{-,0}$ is a lamination of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$. Moreover, it is the central lamination corresponding to the lamination $\mathcal{L}^-$.*

Proof. We show that the equivalence relation $\mathcal{L}^{-,0}$ on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ satisfy properties (1) and (2) of definition 4.0.4 for the local homeomorphism $\amalg$.

Property (1): Let g_1, g_2 be two points in $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$, h_1, h_2 be two points in the intersection $\mathcal{U}_{g_1} \cap \mathcal{U}_{g_2}$ and $p^{+,0}$ be the projection from $\mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \times \mathbb{R}$ onto $\mathcal{U}_{g^+} \times \mathbb{R}$. We see that

$$p^- \circ \amalg_{\mathtt{N}(g_1)}(\mathtt{N}(h_1)) = p^- \circ \amalg_{\mathtt{N}(g_1)}(\mathtt{N}(h_2))$$

if and only if

$$p^- \circ \amalg_{\mathtt{N}(g_2)}(\mathtt{N}(h_1)) = p^- \circ \amalg_{\mathtt{N}(g_2)}(\mathtt{N}(h_2))$$

since we have

$$p^- \circ \amalg_{\mathtt{N}(g_1)}(\mathtt{N}(h_1)) = h_1^- = p^- \circ \amalg_{\mathtt{N}(g_2)}(\mathtt{N}(h_1))$$

and

$$p^- \circ \amalg_{\mathtt{N}(g_1)}(\mathtt{N}(h_2)) = h_2^- = p^- \circ \amalg_{\mathtt{N}(g_2)}(\mathtt{N}(h_2)).$$

Property (2): Let $\{\mathtt{N}(h_i)\}_{i\in\{1,2,..,n\}}$ be a sequence of points such that for all $i \in \{1, 2, .., n-1\}$ we have

$$\mathtt{N}(h_{i+1}) \in \mathcal{U}_{\mathtt{N}(h_i)}$$

and

$$p^- \circ \amalg_{\mathtt{N}(h_i)}(\mathtt{N}(h_i)) = p^- \circ \amalg_{\mathtt{N}(h_i)}(\mathtt{N}(h_{i+1})).$$

Hence for all $i \in \{1, 2, .., n-1\}$ we have $h_i^- = h_{i+1}^-$. Now using proposition 5.2.6 we get that

$$\mathcal{L}^{-,0}_{\mathtt{N}(h_i)} = \mathcal{L}^{-,0}_{\mathtt{N}(h_{i+1})}$$

for all i in $\{1, 2, .., n-1\}$. Hence

$$\mathcal{L}^{-,0}_{\mathtt{N}(h_1)} = \mathcal{L}^{-,0}_{\mathtt{N}(h_n)}.$$

Now we show the other direction. Let $h \in \mathsf{U}_{\text{rec}}\mathbb{H}$ such that $\mathtt{N}(h) \in \mathcal{L}^{-,0}_{\mathtt{N}(g)}$. Using proposition 5.2.6 we get that $h^- = g^-$. Let $\mathcal{V}_{g^+}$ be a connected bounded open neighborhood of g^+ in $\partial_\infty\mathbb{H} \setminus \{g^-\}$ containing the point h^+ and let $\mathcal{V}_{g^-}$ be a connected open neighborhood of g^- in $\partial_\infty\mathbb{H} \setminus \{g^+\}$ such that $\mathcal{V}_{g^+} \cap \mathcal{V}_{g^-}$ is empty. We denote the sets $\mathcal{V}_{g^\pm} \cap \Lambda_\infty\Gamma$ respectively by $\mathcal{U}_{g^\pm}$, the open subset of $\mathsf{U}_{\text{rec}}\mathbb{H}$ corresponding to the open set $\mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \times \mathbb{R}$ by $\mathcal{U}_g$ and the open set $\mathtt{N}(\mathcal{U}_g)$ around $\mathtt{N}(g)$ by $\mathcal{U}_{\mathtt{N}(g)}$. Now we consider the chart $\left(\mathcal{U}_{\mathtt{N}(g)}, \amalg_{\mathtt{N}(g)}\right)$ and notice that

$$p^- \circ \amalg_{\mathtt{N}(g)}(\mathtt{N}(g)) = g^- = h^- = p^- \circ \amalg_{\mathtt{N}(g)}(\mathtt{N}(h)).$$

Therefore we conclude that $\mathcal{L}^{-,0}$ defines a lamination with plaque neighborhoods given by the image of the open sets $\mathcal{U}_{g^+} \times \mathbb{R}$ for g^+ in $\Lambda_\infty\Gamma \setminus \{g^+\}$.

Now the fact that $\mathcal{L}^{-,0}$ is the central lamination corresponding to the lamination $\mathcal{L}^-$ follows from definition 5.2.2. $\square$

Theorem 5.2.10. *The laminations $(\mathcal{L}^+, \mathcal{L}^{-,0})$ and $(\mathcal{L}^-, \mathcal{L}^{+,0})$ define a local product structure on $\mathsf{U}_{\text{rec}}\mathbb{A}$.*

Proof. Using proposition 5.2.7, 5.2.8 and 5.2.9 we get that $(\mathcal{L}^+, \mathcal{L}^{-,0})$ defines a local product structure. In a similar way one can show that $(\mathcal{L}^-, \mathcal{L}^{+,0})$ also defines a local product structure. $\square$

Proposition 5.2.11. *The laminations are equivariant under the action of Γ.*

Proof. Let Z be in $\mathsf{U}_{\text{rec}}\mathbb{A}$ such that $Z = \mathtt{N}(g)$ for some $g \in \mathsf{U}_{\text{rec}}\mathbb{H}$ and $W \in \mathcal{L}^+_Z$. Therefore there exist real numbers s_1, s_2 such that

$$W = (\tilde{N}(g) + s_1\nu^+(g), \nu(g) + s_2\nu^+(g)).$$

Now for all γ in Γ we get,

$$\begin{aligned}\gamma.Z &= \gamma.\mathtt{N}(g)\\ &= \mathtt{N}(\gamma.g)\end{aligned}$$

and

$$\begin{aligned}\gamma.W &= \gamma.(\tilde{N}(g) + s_1\nu^+(g), \nu(g) + s_2\nu^+(g))\\ &= (\gamma.\tilde{N}(g) + s_1.\gamma.\nu^+(g), \gamma.\nu(g) + s_2.\gamma.\nu^+(g))\\ &= (\tilde{N}(\gamma.g) + s_1\nu^+(\gamma.g), \nu(\gamma.g) + s_2\nu^+(\gamma.g)).\end{aligned}$$

Therefore $\gamma.W \in \tilde{\mathcal{L}}^+_{\gamma.Z}$ and $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ is invariant under the action of Γ implies that $\gamma.W \in \mathcal{L}^+_{\gamma.Z}$. Hence we get that for all γ in Γ,

$$\mathcal{L}^+_{\gamma.Z} = \gamma.\mathcal{L}^+_Z.$$

Similarly one can show that for all γ in Γ,

$$\mathcal{L}^-_{\gamma.Z} = \gamma.\mathcal{L}^-_Z.$$

□

Proposition 5.2.12. *The laminations are equivariant under the geodesic flow.*

Proof. Let Z be in $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ such that $Z = \mathtt{N}(g)$ for some $g \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ and $W \in \mathcal{L}^+_Z$. Therefore there exist real numbers s_1, s_2 such that

$$W = (\tilde{N}(g) + s_1\nu^+(g), \nu(g) + s_2\nu^+(g)).$$

We have for all real number t,

$$\begin{aligned}\tilde{\Phi}_t Z &= \tilde{\Phi}_t\mathtt{N}(g)\\ &= (N(g) + t\nu(g), \nu(g))\end{aligned}$$

and

$$\begin{aligned}\tilde{\Phi}_t W &= \tilde{\Phi}_t(N(g) + s_1\nu^+(g), \nu(g) + s_2\nu^+(g))\\ &= (N(g) + s_1\nu^+(g) + t.(\nu(g) + s_2\nu^+(g)), \nu(g) + s_2\nu^+(g))\\ &= ((N(g) + t\nu(g)) + (s_1 + ts_2)\nu^+(g), \nu(g) + s_2\nu^+(g)).\end{aligned}$$

Therefore for all real number t we have $\tilde{\Phi}_t.W \in \tilde{\mathcal{L}}^+_{\tilde{\Phi}_t.Z}$ and $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ is invariant under the geodesic flow implies that $\tilde{\Phi}_t.W \in \mathcal{L}^+_{\tilde{\Phi}_t.Z}$. Hence we get that for all real number t,

$$\mathcal{L}^+_{\tilde{\Phi}_t.Z} = \tilde{\Phi}_t.\mathcal{L}^+_Z.$$

Similarly one can show that for all real number t,

$$\mathcal{L}^-_{\tilde{\Phi}_t.Z} = \tilde{\Phi}_t.\mathcal{L}^-_Z.$$

□

Definition 5.2.13. *We denote the projection of $\mathcal{L}^{\pm}, \mathcal{L}^{\pm,0}$ on the space $\mathsf{U}_{\text{rec}}\mathsf{M}$ respectively by $\underline{\mathcal{L}}^{\pm}$ and $\underline{\mathcal{L}}^{\pm,0}$, where $\mathcal{L}^{\pm}, \mathcal{L}^{\pm,0}$ are as defined in definition 5.2.1.*

Now we define the notion of a leaf lift. The leaf lift is a map from the leaves of the lamination through a point, to the tangent space of $\mathsf{U}\mathbb{A}$ at that point. We will use this leaf lift to compare distance between the metric $\tilde{d}$ and the norm on the tangent space on any point of the leaves. We define the leaf lift as follows:
The *positive leaf lift* is the map,

$$\begin{aligned} i^{+}_{\mathtt{N}(g)} : \tilde{\mathcal{L}}^{+}_{\mathtt{N}(g)} &\longrightarrow \mathsf{T}_{\mathtt{N}(g)}\mathsf{U}\mathbb{A} \\ (N(g) + s_1\nu^{+}(g), \nu(g) + s_2\nu^{+}(g)) &\longmapsto (s_1\nu^{+}(g), s_2\nu^{+}(g)). \end{aligned} \tag{5.2.5}$$

and the *negative leaf lift* is the map,

$$\begin{aligned} i^{-}_{\mathtt{N}(g)} : \tilde{\mathcal{L}}^{-}_{\mathtt{N}(g)} &\longrightarrow \mathsf{T}_{\mathtt{N}(g)}\mathsf{U}\mathbb{A} \\ (N(g) + s_1\nu^{-}(g), \nu(g) + s_2\nu^{-}(g)) &\longmapsto (s_1\nu^{-}(g), s_2\nu^{-}(g)). \end{aligned} \tag{5.2.6}$$

where we identify $\mathsf{T}_{\mathtt{N}(g)}\mathsf{U}\mathbb{A}$ with $\mathsf{T}_{N(g)}\mathbb{A} \times \mathsf{T}_{\nu(g)}\mathsf{S}^1$.

5.3 Contraction Properties

In this section we will prove that the leaves denoted by $\mathcal{L}^{+}$ contracts in the forward direction of the geodesic flow and the leaves denoted by $\mathcal{L}^{-}$ contracts in the backward direction of the geodesic flow. We will prove it only for the forward direction of the flow. The other case will follow similarly. We start with the following construction whose raison d'être would be apparent in proposition 5.3.2.

Proposition 5.3.1. *There exists a Γ-invariant map from $\mathsf{U}_{\text{rec}}\mathbb{A}$ into the space of euclidean metrics on $\mathbb{R}^3 \times \mathbb{R}^3$ sending Z to $\|.\|_Z$ such that for all positive integer n, there exists a positive real number t_n satisfying the following property: if $t > t_n$, $Z \in \mathsf{U}_{\text{rec}}\mathbb{A}$ and $W \in \tilde{\mathcal{L}}^{+}_{Z}$ then*

$$\|i^{+}_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t W) - i^{+}_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t Z)\|_{\tilde{\Phi}_t Z} \leqslant \frac{1}{2^n}\|i^{+}_{Z}(W) - i^{+}_{Z}(Z)\|_{Z}.$$

Proof. Let $\langle|\rangle_{\mathtt{N}(g)}$ be a positive definite bilinear form on the tangent space $\mathsf{T}_{\mathtt{N}(g)}(\mathbb{A} \times \mathbb{V})$ satisfying the following properties,

1. $\langle(\nu^{\alpha}(g), 0)|(\nu^{\beta}(g), 0)\rangle_{\mathtt{N}(g)} = \langle(0, \nu^{\alpha}(g))|(0, \nu^{\beta}(g))\rangle_{\mathtt{N}(g)} = \delta_{\alpha\beta}$,
2. $\langle(\nu^{\alpha}(g), 0)|(0, \nu^{\beta}(g))\rangle_{\mathtt{N}(g)} = \langle(0, \nu^{\alpha}(g))|(\nu^{\beta}(g), 0)\rangle_{\mathtt{N}(g)} = 0$.

where $\delta_{\alpha\beta}$ is the dirac delta function with α, β in $\{., +, -\}$. We define the map $\|.\|$ as follows,

$$\|X\|_{\mathtt{N}(g)} := \sqrt{\langle X|X\rangle_{\mathtt{N}(g)}},$$

where X is in $\mathsf{T}_{\mathtt{N}(g)}(\mathbb{A}\times\mathbb{V})$. Now from equation 2.3.4, equation 2.3.6 and theorem 3.2.1 we get that $\|.\|$ is Γ-invariant, that is,

$$\|\gamma X\|_{\gamma\mathtt{N}(g)} = \|X\|_{\mathtt{N}(g)}.$$

Let $Z = \mathtt{N}(g)$ be in $\mathsf{U}_{\text{rec}}\mathbb{A}$ and $W \in \tilde{\mathcal{L}}^{+}_{\mathrm{Z}}$. Therefore there exists real numbers s_1 and s_2 such that

$$W = (N(g) + s_1\nu^+(g), \nu(g) + s_2\nu^+(g)).$$

Hence the norm is

$$\|i^+_Z(W) - i^+_Z(Z)\|_Z = \|(s_1\nu^+(g), s_2\nu^+(g))\|_Z = \sqrt{s_1^2 + s_2^2}. \tag{5.3.1}$$

We note that $\tilde{\Phi}_t Z = (N(g)+t\nu(g), \nu(g))$ and using theorem 3.2.1 we get that there exists a positive real number t_1 such that

$$N(g) + t\nu(g) = N(\tilde{\phi}_{t_1}g).$$

Moreover t and t_1 are related by the following formula,

$$t = \int_0^{t_1} f(\tilde{\phi}_s g)ds.$$

Therefore we have

$$\begin{aligned}
\|i^+_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t W) - i^+_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t Z)\|_{\tilde{\Phi}_t Z} &= \|((s_1 + ts_2)\nu^+(g), s_2\nu^+(g))\|_{\tilde{\Phi}_t Z}\\
&= \|((s_1 + ts_2)\nu^+(g), s_2\nu^+(g))\|_{\mathtt{N}(\phi_{t_1}g)}\\
&= \sqrt{(s_1 + ts_2)^2 + s_2^2}\ .\ \|(\nu^+(g), 0)\|_{\mathtt{N}(\phi_{t_1}g)}\\
&= \sqrt{(s_1 + ts_2)^2 + s_2^2}\ .\ \|e^{-t_1}(\nu^+(\phi_{t_1}g), 0)\|_{\mathtt{N}(\phi_{t_1}g)}
\end{aligned}$$

Hence the norm is

$$\begin{aligned}
\|i^+_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t W) - i^+_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t Z)\|_{\tilde{\Phi}_t Z} &= \sqrt{(s_1 + ts_2)^2 + s_2^2}\ .\ e^{-t_1}\\
&\leqslant \sqrt{2}\sqrt{s_1^2 + s_2^2}(1+t)e^{-t_1}.
\end{aligned} \tag{5.3.2}$$

We also know that $\mathsf{U}_{\text{rec}}\Sigma$ is compact. Hence f is bounded on $\mathsf{U}_{\text{rec}}\mathbb{H}$. Therefore there exists a constant c_1 such that

$$t = \int_0^{t_1} f(\tilde{\phi}_s g)ds \leqslant \int_0^{t_1} c_1 ds = c_1 t_1.$$

We choose a constant c bigger than $\max\{1, 2c_1\}$ and get that

$$(1+t)e^{-t_1} \leqslant ce^{-\frac{t}{2c_1}}. \tag{5.3.3}$$

Now by combining equation 5.3.1, inequalities 5.3.2 and 5.3.3 we get that

$$\|i^{+}_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t W) - i^{+}_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t Z)\|_{\tilde{\Phi}_t Z} \leqslant \sqrt{2} c e^{-\frac{t}{2c_1}} \|i^{+}_{Z}(W) - i^{+}_{Z}(Z)\|_{Z}.$$

Hence for all positive integer n, there exists $t_n \in \mathbb{R}$ such that if $t > t_n$, $Z \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ and $W \in \mathcal{L}^{+}_{\mathrm{Z}}$ then

$$\|i^{+}_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t W) - i^{+}_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t Z)\|_{\tilde{\Phi}_t Z} \leqslant \frac{1}{2^n} \|i^{+}_{Z}(W) - i^{+}_{Z}(Z)\|_{Z}.$$

$\square$

Proposition 5.3.2. *Let d be a Γ-invariant distance on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ which is locally bilipschitz equivalent to an euclidean distance and let $\|.\|$ be the Γ-invariant map from $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ to the space of euclidean metrics on $\mathbb{R}^3 \times \mathbb{R}^3$ as constructed in the proof of proposition 5.3.1. There exist positive constants K and α such that for any $Z \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ and for any $W \in \mathcal{L}^{+}_{Z}$, the following statements are true,*

1. *If $d(W,Z) \leqslant \alpha$, then $\|i^{+}_{Z}(Z) - i^{+}_{Z}(W)\|_{Z} \leqslant K d(W,Z)$,*
2. *If $\|i^{+}_{Z}(Z) - i^{+}_{Z}(W)\|_{Z} \leqslant \alpha$, then $d(W,Z) \leqslant K \|i^{+}_{Z}(Z) - i^{+}_{Z}(W)\|_{Z}$.*

Proof. Since Γ acts cocompactly on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ and both d and $\|.\|$ are Γ-invariant, it suffices to prove the above assertion for Z in a compact subset D of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$, where D is the closure of a suitably chosen fundamental domain.
We can define an euclidean distance d_Z on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$, uniquely using the euclidean metric $\|.\|_Z$ on $\mathbb{R}^3 \times \mathbb{R}^3$, by taking the embedding of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ in $\mathbb{A} \times \mathbb{R}^3$. We notice that for any Z in $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ and for any W in $\mathcal{L}^{+}_{Z}$, $d_Z(W,Z)$ is equal to $\|i^{+}_{Z}(W) - i^{+}_{Z}(Z)\|_Z$. Now, any two euclidean distances are bilipschitz equivalent with each other and by our hypothesis, d is locally bilipschitz equivant to an euclidean distance. Therefore, in particular, d is locally bilipschitz equivalent with d_Z for Z in D, that is, there exist constants K_Z depending on Z, and open sets U_Z around Z, such that the distance d_Z and d are K_Z bilipschitz equivalent with each other on U_Z.
Let $C_{(X,Y)}$ for any X and Y in D, be a constant such that the distance d_X and d_Y are $C_{(X,Y)}$ bilipschitz equivalent with each other. It follows from the construction of the norm $\|.\|$, as done in proposition 5.3.1, that we can choose the constants $C_{(X,Y)}$ in such a way that $C_{(X,Y)}$ vary continuously on (X,Y). As D is compact it follows that $C_{(X,Y)}$ is bounded above by some constant C. Hence, for all X and Y in D, d_X and d_Y are C bilipschitz equivalent with each other.
Now, we consider the open cover of D by the open sets U_Z. As D is compact, there exist points $Z_1, Z_2, .., Z_n$ in D such that $U_{Z_1}, U_{Z_2}, .., U_{Z_n}$ covers D. Let β be the Lebesgue number of this cover for the distance d and K_0 be the maximum of $K_{Z_1}, K_{Z_2}, .., K_{Z_n}$. Therefore, for any Z in D, the open ball of radius β around Z for the metric d, denoted by $B_d(Z,\beta)$, lies inside U_{Z_j} for some j in $\{1, 2, .., n\}$. Hence, d and d_{Z_j} are K_0 bilipschitz equivalent with each other on $B_d(Z,\beta)$. As d_Z and d_{Z_j} are C bilipschitz equivalent with each other, it follows that d and d_Z are CK_0 bilipschitz equivalent with each other on $B_d(Z,\beta)$. Moreover, we note that the constants β, C, K_0 and hence also CK_0, does not depend on Z. Therefore, d and d_Z are CK_0 bilipschitz equivalent with each other on

$B_d(Z, \beta)$, for all Z in D.
As any two distances d_X and d_Y, for all X, Y in D are C bilipschitz equivalent with each other. Without loss of generality we can choose a point X in D and consider the distance d_X. The note that the set $\{B_d(Z, \beta) : Z \in D\}$ is an open cover of D. Let β_1 be a Lebesgue number for this cover for the metric space (D, d_X). Therefore, the open ball $B_{d_X}(Y_1, \beta_1)$ for any Y_1 in D, lies inside an open ball $B_d(Y_2, \beta)$ for some point Y_2 in D. Now, as d and d_Z are CK_0 bilipschitz equivalent with each other on the ball $B_d(Z, \beta)$ for all Z in D, it follows that d and d_X are CK_0 bilipschitz equivalent with each other on the ball $B_{d_X}(Y_2, \beta_1)$. As Y_2 was chosen arbitrarily we have that d and d_X are CK_0 bilipschitz equivalent with each other on the ball $B_{d_X}(Y, \beta_1)$, for all Y in D.
Now, we know that d_X and d_Z are C bilipschitz equivalent with each other. Therefore we get that d and d_Z are CK_0 bilipschitz equivalent with each other on the ball $B_{d_Z}(Y, \frac{\beta_1}{C})$, for all Y in D. In particular one has, d and d_Z are CK_0 bilipschitz equivalent with each other on the ball $B_{d_Z}(Z, \frac{\beta_1}{C})$. Finally, set α to be $\min\{\frac{\beta_1}{C}, \beta\}$ and K to be CK_0 to get that for any Z in $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ and W in $\mathcal{L}^+_{\mathrm{Z}}$ we have,

1. If $d(W, Z) \leqslant \alpha,$ then $\|i^+_Z(Z) - i^+_Z(W)\|_Z \leqslant K d(W, Z)$,
2. If $\|i^+_Z(Z) - i^+_Z(W)\|_Z \leqslant \alpha,$ then $d(W, Z) \leqslant K\|i^+_Z(Z) - i^+_Z(W)\|_Z$.

. □

Theorem 5.3.3. *Let $\mathcal{L}^{\pm}$ be two laminations on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ as defined in definitions 5.2.1, 5.2.2 and let $\tilde{d}$ be the Γ invariant metric, as defined in definition 5.1.2. Under these assumptions, for the metric $\tilde{d}$ on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ we have that*

1. *$\mathcal{L}^+$ is contracted in the forward direction of the geodesic flow, and,*
2. *$\mathcal{L}^-$ is contracted in the backward direction of the geodesic flow.*

Proof. Let $\|.\|$ be the Γ-invariant map from $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ to the space of euclidean metrics on $\mathbb{R}^3 \times \mathbb{R}^3$ as constructed in the proof of proposition 5.3.1 and let K and α be as in the proposition 5.3.2 for the distance $\tilde{d}$. We choose a positive integer n such that

$$\frac{K}{2^n} < 1 \ , \ \frac{K^2}{2^n} \leqslant \frac{1}{2}.$$

Let t_n be the constant as in proposition 5.3.1 for our chosen n. Also let Z be in $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ and W be in $\mathcal{L}^+_Z$, so that $\tilde{d}(W, Z) \leqslant \alpha$. Then using proposition 5.3.2 we get

$$\|i^+_Z(W) - i^+_Z(Z)\|_Z \leqslant K\tilde{d}(W, Z).$$

Furthermore, using proposition 5.3.1 we get for all $t > t_n$ that

$$\|i^+_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t W) - i^+_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t Z)\|_{\tilde{\Phi}_t Z} \leqslant \frac{1}{2^n}\|i^+_Z(W) - i^+_Z(Z)\|_Z.$$

It follows that

$$\|i^+_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t W) - i^+_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t Z)\|_{\tilde{\Phi}_t Z} \leqslant \frac{K\alpha}{2^n} \leqslant \alpha.$$

Hence again using proposition 5.3.2 we have

$$\tilde{d}(\tilde{\Phi}_t W, \tilde{\Phi}_t Z) \leqslant K \| i^+_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t W) - i^+_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t Z) \|_{\tilde{\Phi}_t Z}.$$

Combining the above inequalities, for all $t > t_n$ we get

$$\tilde{d}(\tilde{\Phi}_t W, \tilde{\Phi}_t Z) \leqslant \frac{K^2}{2^n} \tilde{d}(W, Z) \leqslant \frac{1}{2}\, \tilde{d}(W, Z). \tag{5.3.4}$$

Hence $\mathcal{L}^+$ is contracted in the forward direction of the geodesic flow. The proof of the contraction of $\mathcal{L}^-$ follows similarly. $\square$

5.4 Metric Anosov structure on the quotient

Let us now consider what happens in the quotient, that is, $\mathsf{U}_{\text{rec}}\mathsf{M}$. Let Z be in $\mathsf{U}_{\text{rec}}\mathbb{A}$ and ϵ be a positive real number. We define,

$$\mathcal{L}^{\pm}_{\epsilon}(Z) := \mathcal{L}^{\pm}_{Z} \cap B_{\tilde{d}}(Z, \epsilon),$$

and

$$\mathcal{K}_\epsilon(Z) := \Pi_Z \left(\mathcal{L}^+_\epsilon(Z) \times \mathcal{L}^-_\epsilon(Z) \times (-\epsilon, \epsilon) \right) \subset \mathsf{U}_{\text{rec}}\mathbb{A}$$

where Π_Z is the local product structure at Z defined by the stable and unstable leaves.

We know that there exists a positive real number ϵ_0 such that for any non identity element γ of Γ and for Z in $\mathsf{U}_{\text{rec}}\mathbb{A}$ we have,

$$\gamma(\mathcal{K}_{\epsilon_0}(Z)) \cap \mathcal{K}_{\epsilon_0}(Z) = \emptyset.$$

Proof of Theorem 0.0.40. Let us fix α as in proposition 5.3.2 and let ϵ_1 be from the open interval $\left(0, \min\left\{\alpha, \frac{\epsilon_0}{2}\right\}\right)$. Now let z be any point of $\mathsf{U}_{\text{rec}}\mathsf{M}$ and let Z be a point in $\mathsf{U}_{\text{rec}}\mathbb{A}$ in the preimage of z. Our choice of ϵ_1 gives us that the inequality 5.3.4 holds for the geodesic flow on $\mathsf{U}_{\text{rec}}\mathbb{A}$ for the points in the chart $\mathcal{K}_{\epsilon_1}(Z)$. Hence the inequality 5.3.4 also holds for the geodesic flow on $\mathsf{U}_{\text{rec}}\mathsf{M}$ for points in the chart which is in the projection of $\mathcal{K}_{\epsilon_1}(Z)$.

Therefore $\underline{\mathcal{L}}^+$, the projection of $\mathcal{L}^+$ in $\mathsf{U}_{\text{rec}}\mathsf{M}$, is contracted in the forward direction of the geodesic flow on $\mathsf{U}_{\text{rec}}\mathsf{M}$. A similar proof holds for $\underline{\mathcal{L}}^-$, the projection of $\mathcal{L}^-$ in $\mathsf{U}_{\text{rec}}\mathsf{M}$, too. $\square$

Chapter 6

Anosov representations

In this chapter we define the notion of an Anosov representation in the context of the non-semisimple Lie group $\mathsf{G} := \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$.

6.1 Pseudo-Parabolic subgroups

Let $\mathbb{X}$ be the space of all affine null planes. We observe that G acts transitively on $\mathbb{X}$. Hence for all $P \in \mathbb{X}$ we have

$$\mathbb{X} = \mathsf{G}.P \cong \mathsf{G}/\mathsf{Stab}_{\mathsf{G}}(P).$$

Definition 6.1.1. *If $P \in \mathbb{X}$ then we define*

$$\mathsf{P}_P := \mathsf{Stab}_{\mathsf{G}}(P).$$

We call P_P a pseudo-parabolic subgroup of G.

Let $\mathsf{V}(P)$ denote the vector space underlying a null plane P, let $v_0 := (1,0,0)^t$ and $v_0^{\pm} := (0,\pm1,1)^t$ and let $\mathcal{C}$ be the upper half of $\mathsf{S}^0\backslash\{0\}$. Now we consider the space

$$\mathcal{N} := \{(P_1,P_2) \mid P_1,P_2 \in \mathbb{X}, \mathsf{V}(P_1) \neq \mathsf{V}(P_2)\}$$

and define the following map

$$\begin{aligned} v : \mathcal{N} &\longrightarrow \mathsf{S}^1 \\ (P_1,P_2) &\longmapsto v(P_1,P_2) \end{aligned}$$

where $v(P_1,P_2) \in \mathsf{V}(P_1) \cap \mathsf{V}(P_2) \cap \mathsf{S}^1$ is such that if $v(Q_1) \in \mathsf{V}(Q_1) \cap \mathcal{C}$ and $v(Q_2) \in \mathsf{V}(Q_2) \cap \mathcal{C}$ then $(v(Q_1), v(Q_1,Q_2), v(Q_2))$ gives the same orientation as (v_0^+, v_0, v_0^-). We observe that

$$v(P_1,P_2) = -v(P_2,P_1).$$

Proposition 6.1.2. *The space $\mathcal{N}$ is the unique open G orbit in $\mathbb{X} \times \mathbb{X}$ for the diagonal action of G on $\mathbb{X} \times \mathbb{X}$.*

Proof. We start by observing that $\mathcal{N}$ is open and dense in $\mathbb{X} \times \mathbb{X}$. Now let (P_1, P_2) and (Q_1, Q_2) be two arbitrary points in $\mathcal{N}$. We consider the vector $v(P_1, P_2) \in \mathsf{S}^1$ corresponding to the point (P_1, P_2) and the vector $v(Q_1, Q_2) \in \mathsf{S}^1$ corresponding to the point (Q_1, Q_2). Now as $\mathsf{SO}^0(2,1)$ acts transitively on S^1 we get that there exist $g \in \mathsf{SO}^0(2,1)$ such that

$$v(Q_1, Q_2) = g.v(P_1, P_2).$$

We choose $X(Q_1, Q_2) \in Q_1 \cap Q_2$ and $X(P_1, P_2) \in P_1 \cap P_2$ and observe that

$$(e, X(Q_1, Q_2) - O) \circ (g, 0) \circ (e, X(P_1, P_2) - O)^{-1}.P_1 = Q_1,$$

$$(e, X(Q_1, Q_2) - O) \circ (g, 0) \circ (e, X(P_1, P_2) - O)^{-1}.P_2 = Q_2,$$

where e is the identity element in $\mathsf{SO}^0(2,1)$. Therefore $\mathcal{N}$ is an open G orbit in $\mathbb{X} \times \mathbb{X}$. Now as $\mathbb{X} \times \mathbb{X}$ is connected the result follows. $\square$

Let N be the space of oriented space like affine lines. We think of N as the space $\mathsf{U}\mathbb{A}/\sim$ where $(X, v) \sim (X_1, v_1)$ if and only if $(X_1, v_1) = \tilde{\Phi}_t(X, v)$ for some $t \in \mathbb{R}$. We denote the equivalence class of (X, v) by $[(X, v)]$. Now let us consider the following map

$$\begin{aligned} \imath' : \mathcal{N} &\longrightarrow \mathsf{N} \\ (P_1, P_2) &\longmapsto [(X(P_1, P_2), v(P_1, P_2))] \end{aligned}$$

where $X(P_1, P_2)$ is any point in $P_1 \cap P_2$. We observe that $\imath'$ gives a G equivariant map.

Let us denote the plane passing through X with underlying vector space generated by the vectors w_1 and w_2 by P_{X,w_1,w_2}. Now we consider another map

$$\begin{aligned} \imath : \mathsf{U}\mathbb{A} &\longrightarrow \mathcal{N} \\ (X, v) &\longmapsto (P_{X,v,v^+}, P_{X,v,v^-}) \end{aligned}$$

where $v^\pm \in \mathcal{C}$ such that $\langle v^\pm \mid v \rangle = 0$ and (v^+, v, v^-) gives the same orientation as (v_0^+, v_0, v_0^-). We observe that $\imath$ is a G equivariant map. Now as $P_{X+tv,v,v^+} = P_{X,v,v^+}$ and $P_{X+tv,v,v^-} = P_{X,v,v^-}$ we get that the map $\imath$ gives rise to a map, which we again denote by $\imath$,

$$\imath : \mathsf{N} \longrightarrow \mathcal{N}.$$

Moreover, we observe that $\imath \circ \imath' = \mathsf{Id}$ and $\imath' \circ \imath = \mathsf{Id}$.

6.2 Geometric Anosov structure

Geometric Anosov structures were first intoduced by Labourie in [25]. In this section we give an appropriate definition of geometric Anosov property and show that $(\mathsf{U}_{\text{rec}}\mathsf{M}, \mathcal{L})$ admits a geometric Anosov structure.

Let $(P^+, P^-) \in \mathcal{N}$ such that $P^+ := P_{O,v_0,v_0^+}$ and $P^- := P_{O,v_0,v_0^-}$. We denote $\mathsf{Stab}_\mathsf{G}(P^\pm)$ respectively by $\mathsf{P}^\pm$. We note that the pair $\mathbb{X}^\pm := \mathsf{G}/\mathsf{P}^\pm$ gives a pair of continuous foliations on the space N whose tangential distributions $\mathsf{E}^\pm$ satisfy

$$\mathsf{TN} = \mathsf{E}^+ \oplus \mathsf{E}^-.$$

Definition 6.2.1. *We say that a vector bundle E over a compact topological space whose total space is equipped with a flow $\{\varphi_t\}_{t\in\mathbb{R}}$ of bundle automorphisms is contracted by the flow as $t \to \infty$ if and only if for any metric $\|.\|$ on E, there exists positive constants t_0, A and c such that for all $t > t_0$ and for all v in E we have*

$$\|\varphi_t(v)\| \leqslant Ae^{-ct}\|v\|.$$

Definition 6.2.2. *Let $\mathcal{L}$ denote the orbit foliation of $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ under the flow Φ. We say that $(\mathsf{U}_{\mathrm{rec}}\mathsf{M}, \mathcal{L})$ admits a geometric (N, G)-Anosov structure if and only if there exist a map*

$$F : \widetilde{\mathsf{U}_{\mathrm{rec}}\mathsf{M}} \longrightarrow \mathsf{N}$$

such that the following holds:

1. *For all $\gamma \in \Gamma$ we have $F \circ \gamma = \gamma \circ F$,*
2. *For all $t \in \mathbb{R}$ we have $F \circ \tilde{\Phi}_t = F$,*
3. *By the flow invariance, the bundles $F^{\pm} := F^*\mathsf{E}^{\pm}$ are equipped with a parallel transport along the orbits of $\tilde{\Phi}$. The bundle F^+ gets contracted by the lift of the flow $\tilde{\Phi}_t$ as $t \to \infty$ and F^- gets contracted by the lift of the flow $\tilde{\Phi}_t$ as $t \to -\infty$.*

Proof of Theorem 0.0.41. Let us define the map F as follows:

$$\begin{aligned} F : \widetilde{\mathsf{U}_{\mathrm{rec}}\mathsf{M}} &\longrightarrow \mathsf{N} \\ (X, v) &\longmapsto [(X, v)] \end{aligned}$$

We note that the map F is clearly Γ-equivariant and is also invariant under the flow $\tilde{\Phi}$. Now we observe that

$$\mathsf{T}_{\imath([(X,v)])}\mathsf{G}/\mathsf{P}^- \cong \mathbb{R}.v^+ \oplus \mathbb{R}.v^+$$

and

$$\mathsf{T}_{\imath([(X,v)])}\mathsf{G}/\mathsf{P}^+ \cong \mathbb{R}.v^- \oplus \mathbb{R}.v^-$$

where $v^+, v^- \in \mathcal{C}$ such that $\langle v^{\pm} \mid v\rangle = 0$ and (v^+, v, v^-) gives the same orientation as (v_0^+, v_0, v_0^-).

Now using proposition 5.3.1 we notice that F^+ gets contracted by the lift of the flow $\tilde{\Phi}_t$ as $t \to \infty$ and F^- gets contracted by the lift of the flow $\tilde{\Phi}_t$ as $t \to -\infty$. Moreover, as $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ is compact we have that the convergence is independent of the choice of the metric. $\square$

6.3 Gromov geodesic flow

Now let $\partial_\infty\Gamma$ be the Gromov boundary of the free group Γ. Also let

$$\partial_\infty\Gamma^{(2)} := \partial_\infty\Gamma \times \partial_\infty\Gamma \setminus \{(x, x) \mid x \in \partial_\infty\Gamma\}.$$

Let $\mathbb{R}$ acts on $\widetilde{\mathsf{U}_0\Gamma} := \partial_\infty\Gamma^{(2)} \times \mathbb{R}$ by translation on the last factor. In [20] Gromov defined a proper cocompact action of Γ on $\partial_\infty\Gamma^{(2)} \times \mathbb{R}$ which commutes with the action of $\mathbb{R}$. The restriction of this action on $\partial_\infty\Gamma^{(2)}$ is the diagonal action coming from the standard action of Γ on $\partial_\infty\Gamma$. There is a metric on $\widetilde{\mathsf{U}_0\Gamma}$ such that the Γ action is isometric. The metric is well defined up to Hölder equivalence. Moreover, every orbit of the $\mathbb{R}$ action gives a quasi-isometric embedding and the geodesic flow $\tilde{\psi}_t$ acts by Lipschitz homeomorphisms. The flow $\tilde{\psi}_t$ on $\widetilde{\mathsf{U}_0\Gamma}$ descends to a flow ψ_t on the quotient $\mathsf{U}_0\Gamma := \Gamma\backslash\left(\partial_\infty\Gamma^{(2)} \times \mathbb{R}\right)$. We call it the Gromov geodesic flow. We denote the projection onto the first coordinate of $\widetilde{\mathsf{U}_0\Gamma}$ by π_1 and the projection onto the second coordinate of $\widetilde{\mathsf{U}_0\Gamma}$ by π_2. More details about this construction can be found in Champetier [10] and Mineyev [32].

6.4 Anosov structure

Let $\mathsf{G} = \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$ and let

$$\mathsf{P}^\pm := \mathsf{Stab}_\mathsf{G}(P_{O,v_0,v_0^\pm}).$$

Also let $\mathsf{L} = \mathsf{P}^+ \cap \mathsf{P}^-$. We note that $\mathsf{L} = \mathsf{Stab}_\mathsf{G}([\mathsf{P}^+],[\mathsf{P}^-])$ for the diagonal action of G on $\mathsf{G}/\mathsf{P}^+ \times \mathsf{G}/\mathsf{P}^-$. Moreover, using proposition 6.1.2 we get that the G orbit of the point $([\mathsf{P}^+],[\mathsf{P}^-]) \in \mathsf{G}/\mathsf{P}^+ \times \mathsf{G}/\mathsf{P}^-$ is the unique open G orbit in $\mathsf{G}/\mathsf{P}^+ \times \mathsf{G}/\mathsf{P}^-$. We also note that

$$\mathsf{G}/\mathsf{L} = \mathsf{G}.([\mathsf{P}^+],[\mathsf{P}^-]).$$

Moreover, the pair $\mathsf{G}/\mathsf{P}^\pm$ gives a continuous set of foliations on the space G/L whose tangential distributions $\mathsf{E}^\pm$ satisfy

$$\mathsf{T}(\mathsf{G}/\mathsf{L}) = \mathsf{E}^+ \oplus \mathsf{E}^-.$$

We denote the Lie algebras associated to the Lie groups $\mathsf{G}, \mathsf{P}^\pm$ and L respectively by $\mathfrak{g}, \mathfrak{p}^\pm$ and $\mathfrak{l}$. We notice that

$$\mathfrak{g} = \mathfrak{p}^+ + \mathfrak{p}^- \quad \text{and} \quad \mathfrak{l} = \mathfrak{p}^+ \cap \mathfrak{p}^-. \tag{6.4.1}$$

If we complexify, we obtain the Lie algebras $\mathfrak{p}^\pm_\mathbb{C}$ and $\mathfrak{l}_\mathbb{C}$, so that the same equation 6.4.1 is satisfied, that is,

$$\mathfrak{g}_\mathbb{C} = \mathfrak{p}^+_\mathbb{C} + \mathfrak{p}^-_\mathbb{C} \quad \text{and} \quad \mathfrak{l}_\mathbb{C} = \mathfrak{p}^+_\mathbb{C} \cap \mathfrak{p}^-_\mathbb{C}. \tag{6.4.2}$$

Now as $\mathsf{SO}^0(2,1)$ is a subgroup of $\mathsf{GL}(\mathbb{R}^3)$ we get

$$\mathsf{G}_\mathbb{C} = \mathsf{SO}(3,\mathbb{C}) \ltimes \mathbb{C}^3.$$

We call a complex plane P *degenerate* if and only if there exist a non zero vector $(v_1, v_2, v_3)^t \in P$ such that for all $(v_1', v_2', v_3')^t \in P$ we have

$$v_1v_1' + v_2v_2' + v_3v_3' = 0.$$

Let us denote the space of all complex degenerate planes by $\mathbb{Y}_\mathbb{C}$. The group $\mathsf{SO}(3,\mathbb{C})$ acts transitively on the space $\mathbb{Y}_\mathbb{C}$. Moreover, the action of the group $\mathsf{SO}(3,\mathbb{C})$ is transitive on the following space:

$$\mathbb{Y}^{(2)}_\mathbb{C} := \{(P_1,P_2) \in \mathbb{Y}_\mathbb{C} \times \mathbb{Y}_\mathbb{C} \mid P_1 \neq P_2\}.$$

Now let $\mathbb{X}_\mathbb{C}$ be the space of all affine degenerate planes in $\mathbb{C}^3$. We consider the following open subspace:

$$\mathcal{N}_\mathbb{C} := \{(P_1, P_2) \in \mathbb{X}_\mathbb{C} \times \mathbb{X}_\mathbb{C} \mid \mathsf{V}(P_1) \neq \mathsf{V}(P_2)\}$$

and using the fact that $\mathsf{SO}(3,\mathbb{C})$ acts transitively on the space $\mathbb{Y}_\mathbb{C}^{(2)}$, we deduce that the action of the group $\mathsf{G}_\mathbb{C} = \mathsf{SO}(3,\mathbb{C}) \ltimes \mathbb{C}^3$ on the space $\mathcal{N}_\mathbb{C}$ is transitive. Moreover, we fix $(P_1, P_2) \in \mathcal{N}_\mathbb{C}$ and observe that

$$\mathsf{L}_\mathbb{C} \cong \mathsf{Stab}_{\mathsf{G}_\mathbb{C}}(P_1, P_2)$$

where $\mathsf{L}_\mathbb{C}$ denote the complexification of the group L. Hence

$$\mathsf{G}_\mathbb{C}/\mathsf{L}_\mathbb{C} \cong \mathcal{N}_\mathbb{C}.$$

Now using equation 6.4.2 we get that $\mathsf{G}_\mathbb{C}/\mathsf{L}_\mathbb{C}$ is foliated by two foliations, whose stabilizers are $\mathsf{P}^\pm_\mathbb{C}$ respectively. We denote the tangential distributions corresponding to the foliations $\mathsf{G}_\mathbb{C}/\mathsf{P}^\pm_\mathbb{C}$ respectively by $\mathsf{E}^\pm_\mathbb{C}$ and observe that

$$\mathsf{T}(\mathsf{G}_\mathbb{C}/\mathsf{L}_\mathbb{C}) = \mathsf{E}^+_\mathbb{C} \oplus \mathsf{E}^-_\mathbb{C}.$$

Definition 6.4.1. *We say that ρ in $\mathsf{Hom}(\Gamma, \mathsf{G})$ (respectively $\mathsf{Hom}(\Gamma, \mathsf{G}_\mathbb{C})$) is $(\mathsf{G}, \mathsf{P}^\pm)$-Anosov (respectively $(\mathsf{G}_\mathbb{C}, \mathsf{P}^\pm_\mathbb{C})$-Anosov) if and only if there exist two continuous maps*

$$\xi^\pm_\rho : \partial_\infty\Gamma \longrightarrow \mathsf{G}/\mathsf{P}^\pm \text{ (respectively } \mathsf{G}_\mathbb{C}/\mathsf{P}^\pm_\mathbb{C})$$

such that the following conditions hold:

1. *For all γ in Γ we have $\xi^\pm_\rho \circ \gamma = \rho(\gamma).\xi^\pm_\rho$.*
2. *If $x \neq y$ in $\partial_\infty\Gamma$ then $(\xi^+_\rho(x), \xi^-_\rho(y))$ lies in G/L (respectively $\mathsf{G}_\mathbb{C}/\mathsf{L}_\mathbb{C}$).*
3. *The induced bundle $\Xi^+_\rho := (\xi^+_\rho \circ \pi_1)^*\mathsf{E}^+$ (respectively $(\xi^+_\rho \circ \pi_1)^*\mathsf{E}^+_\mathbb{C}$) gets contracted by the lift of the flow $\tilde{\psi}_t$ as $t \to \infty$, and the induced bundle $\Xi^-_\rho := (\xi^-_\rho \circ \pi_2)^*\mathsf{E}^-$ (respectively $(\xi^-_\rho \circ \pi_2)^*\mathsf{E}^-_\mathbb{C}$) gets contracted by the lift of the flow $\tilde{\psi}_t$ as $t \to -\infty$.*

The maps $\xi^\pm_\rho$ are called the limit maps associated with the $(\mathsf{G}, \mathsf{P}^\pm)$-Anosov (respectively $(\mathsf{G}_\mathbb{C}, \mathsf{P}^\pm_\mathbb{C})$-Anosov) representation ρ.

The notion of an Anosov representation was first introduced by Labourie in [25]. Furthermore, Guichard–Wienhard studied Anosov representations in more details in [21].

Proposition 6.4.2. *If ρ is in $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ then ρ is $(\mathsf{G}, \mathsf{P}^\pm)$-Anosov.*

Proof. Let $(X, v) \in \mathsf{U}\mathbb{A}$. Let $v^\perp$ be the plane which is perpendicular to the vector v in the Lorentzian metric. We note that $v^\perp \cap \mathcal{C}$ is a disjoint union of two half lines where $\mathcal{C}$ is the upper half of $\mathsf{S}^0\setminus\{0\}$. We choose $v^\pm \in v^\perp \cap \mathcal{C}$ such that (v^+, v, v^-) gives the same orientation as (v_0^+, v_0, v_0^-). Let $P_{X,v,v^\pm}$ respectively be the affine null plane passing through X such that its underlying vector space is generated by v and $v^\pm$. We notice that $P_{X,v,v^+} \neq P_{X,v,v^-}$. Now using proposition 6.1.2 we get that there exist $g_{(X,v)} \in \mathsf{G}$ such that

$$g_{(X,v)}.P_{O,v_0,v_0^+} = P_{X,v,v^+}$$

and

$$g_{(X,v)}.P_{O,v_0,v_0^-} = P_{X,v,v^-}.$$

Moreover, if $g_1 \in \mathsf{G}$ such that

$$g_1.P_{O,v_0,v_0^+} = P_{X,v,v^+}$$

then $g_1^{-1}.g_{(X,v)}$ stabilizes the plane P_{O,v_0,v_0^+}. Hence $g_1^{-1}.g_{(X,v)} \in \mathsf{P}^+$. Therefore the following is a well defined map:

$$\begin{aligned} \eta^+ : \mathsf{U}\mathbb{A} &\longrightarrow \mathsf{G}/\mathsf{P}^+ \\ (X,v) &\longmapsto [g_{(X,v)}.\mathsf{P}^+]. \end{aligned}$$

We notice that η^+ is G-equivariant. Similarly, we define another G-equivariant map

$$\begin{aligned} \eta^- : \mathsf{U}\mathbb{A} &\longrightarrow \mathsf{G}/\mathsf{P}^- \\ (X,v) &\longmapsto [g_{(X,v)}.\mathsf{P}^-]. \end{aligned}$$

Moreover, for all $(X,v) \in \mathsf{U}\mathbb{A}$ we see that

$$(\eta^+,\eta^-)(X,v) = ([g_{(X,v)}.\mathsf{P}^+],[g_{(X,v)}.\mathsf{P}^-]) = g_{(X,v)}.([\mathsf{P}^+],[\mathsf{P}^-]).$$

Hence $(\eta^+,\eta^-)(\mathsf{U}\mathbb{A}) \subset \mathsf{G}/\mathsf{L}$.

Now let $\rho \in \mathsf{Hom}_{\mathrm{M}}(\Gamma,\mathsf{G})$. Hence $\mathsf{L}_\rho \in \mathsf{Hom}_{\mathrm{S}}(\Gamma,\mathsf{SO}^0(2,1))$. Now Γ being a free group we get that there exists a Γ-equivariant homeomorphism

$$\iota_\rho : \partial_\infty\Gamma \longrightarrow \Lambda_\infty \mathsf{L}_\rho(\Gamma).$$

We define

$$\eta^\pm_\rho := \eta^\pm\big|_{\mathsf{U}^\rho_{\mathrm{rec}}\mathbb{A}}$$

and observe that for any $[g.\mathsf{P}^+] \in \mathsf{G}/\mathsf{P}^+$ we have

$$(\eta^+_\rho)^{-1}\left([g.\mathsf{P}^+]\right) = \{(X,v) \mid (X,v) \in \mathcal{L}^{+,0}_{(g.O,\mathsf{L}(g)v_0)}\}$$

Now using proposition 5.2.6 we notice that the maps $\eta^\pm_\rho \circ \mathtt{N}_\rho$ gives rise to a pair of Γ-equivariant continuous maps

$$\zeta^\pm_\rho : \Lambda_\infty \mathsf{L}_\rho(\Gamma) \longrightarrow \mathsf{G}/\mathsf{P}^\pm.$$

Therefore the following map,

$$\xi^\pm_\rho := \zeta^\pm_\rho \circ \iota_\rho : \partial_\infty\Gamma \longrightarrow \mathsf{G}/\mathsf{P}^\pm$$

is also continuous and Γ-equivariant. Moreover, as $(\eta^+_\rho,\eta^-_\rho)(\mathsf{U}^\rho_{\mathrm{rec}}\mathbb{A}) \subset \mathsf{G}/\mathsf{L}$ we get that if $x,y \in \partial_\infty\Gamma$ with $x \neq y$ then $(\zeta^+_\rho(x),\zeta^-_\rho(y)) \in \mathsf{G}/\mathsf{L}$. We also observe that

$$\mathsf{T}_{[g.\mathsf{P}^\pm]}\mathsf{G}/\mathsf{P}^\pm \cong \mathbb{R}.\mathtt{L}(g)v_0^\mp \oplus \mathbb{R}.\mathtt{L}(g)v_0^\mp.$$

Now using proposition 5.3.1 we conclude that ρ is $(\mathsf{G},\mathsf{P}^\pm)$-Anosov. $\square$

Chapter 7

Thermodynamical formalism

In this chapter, we describe the theory of thermodynamical formalism as appeared in [8]. We include this chapter for the sake of completeness. The theory had been originally developed by Bowen, Parry–Pollicott, Ruelle and others. We also describe a variation of a construction of McMullen, which produces a pressure form on the space of pressure zero functions on a flow space.

7.1 Hölder flows

Let $\mathcal{X}$ be a compact metric space with a Hölder continuous flow $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ without fixed points.

7.1.1 Reparametrizations

Let $f : \mathcal{X} \to \mathbb{R}$ be a positive Hölder continuous function. Now as $\mathcal{X}$ is compact, f has a positive minimum and for all $x \in \mathcal{X}$, the function $\kappa_f : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$, defined by

$$\kappa_f(x,t) := \int_0^t f(\phi_s x) ds,$$

is an increasing homeomorphism of $\mathbb{R}$. Hence we have a map $\alpha_f : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ such that

$$\alpha_f(x, \kappa_f(x,t)) = \kappa_f(x, \alpha_f(x,t)) = t,$$

for all $(x,t) \in \mathcal{X} \times \mathbb{R}$. The reparametrization of the flow ϕ by f is denoted by the flow $\phi^f = \{\phi^f_t\}_{t\in\mathbb{R}}$ on $\mathcal{X}$ and is defined as follows:

$$\phi^f_t(x) := \phi_{\alpha_f(x,t)}(x),$$

where $t \in \mathbb{R}$ and $x \in \mathcal{X}$.

7.1.2 Livšic-cohomology

Definition 7.1.1. *Let $f, g : \mathcal{X} \to \mathbb{R}$ be two Hölder continuous functions. We say that f is Livšic-cohomologous to g if and only if there exists a function $V : \mathcal{X} \to \mathbb{R}$ such that V is C^1 in the direction of the flow and for all $x \in \mathcal{X}$*

$$f(x) - g(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} V(\phi_t x).$$

We note that:

1. If f and g are Liv̌sic cohomologous then they have the same integral over any ϕ-invariant measure, and
2. If f and g are both positive and Liv̌sic cohomologous, then the flows ϕ^f and ϕ^g are Hölder conjugate.

7.1.3 Periods and measures

Let us denote the set of all periodic orbits of ϕ by O. Now if $a \in \mathsf{O}$ then its period as a ϕ^f periodic orbit is

$$\int_0^{p(a)} f(\phi_s(x))ds$$

where $p(a)$ is the period of a for the flow ϕ and $x \in a$. In particular, if $\widehat{\delta}_a$ is the probability measure invariant under the flow and supported by the orbit a, and if

$$\widehat{\delta}_a = \frac{\delta_a}{\langle \delta_a \mid 1 \rangle},$$

then we have that

$$\langle \delta_a \mid f \rangle = \int_0^{p(a)} f(\phi_s(x))ds$$

and $p(a) = \langle \delta_a \mid 1 \rangle$. In general, if $\mathfrak{m}$ is a ϕ-invariant measure on $\mathcal{X}$ and $f : \mathcal{X} \to \mathbb{R}$ is a Hölder continuous function, then let us use the following notation:

$$\langle \mathfrak{m} \mid f \rangle := \int_{\mathcal{X}} f \mathsf{d}\mathfrak{m}.$$

Now let $\mathfrak{m}$ be a ϕ-invariant probability measure on $\mathcal{X}$ and let ϕ^f be the reparametrization of ϕ by f. We define $\widehat{f.\mathfrak{m}}$ as follows:

$$\widehat{f.\mathfrak{m}} := \frac{1}{\langle \mathfrak{m} \mid f \rangle} f.\mathfrak{m}.$$

We notice that the map $\mathfrak{m} \mapsto \widehat{f.\mathfrak{m}}$ gives a bijection between ϕ-invariant probability measures and ϕ^f-invariant probability measures. In fact if $\widehat{\delta}_a^f$ is the unique ϕ^f invariant probability measure supported by a, then

$$\widehat{\delta}_a^f = \widehat{f.\delta_a}.$$

Hence we have that

$$\langle \widehat{\delta}_a^f \mid g \rangle = \frac{\langle \delta_a \mid f.g \rangle}{\langle \delta_a \mid f \rangle}. \tag{7.1.1}$$

7.1.4 Entropy and pressure

Let $\mathfrak{m}$ be a ϕ-invariant probability measure on $\mathcal{X}$ and let $h(\phi, \mathfrak{m})$ be its metric entropy. Now we describe a relation between the metric entropies of a flow and its reparameterization as follows:

$$h(\phi^f, \widehat{f.\mathfrak{m}}) = \frac{h(\phi, \mathfrak{m})}{\int f \mathsf{d}\mathfrak{m}}.$$

We note that the above equation is called the *Abramov formula*. Now let $\mho^\phi$ be the set of all ϕ-invariant probability measures.

Definition 7.1.2. *We define the pressure of a function $f : \mathcal{X} \to \mathbb{R}$ as follows:*

$$\wp(\phi, f) := \sup_{\mathfrak{m} \in \mho^\phi} \left(h(\phi, \mathfrak{m}) + \int_{\mathcal{X}} f \mathsf{d}\mathfrak{m} \right).$$

In particular,

$$h_{top}(\phi) := \wp(\phi, 0)$$

is called the topological entropy of the flow ϕ.

We note that the pressure $\wp(\phi, f)$ only depends on the Livšic cohomology class of f. We say that a measure $\mathfrak{m} \in \mho^\phi$ on $\mathcal{X}$ is an *equilibrium state* of f if and only if the following equation holds:

$$\wp(\phi, f) = h(\phi, m) + \int_{\mathcal{X}} f \mathsf{d}\mathfrak{m}.$$

An equilibrium state for the function $f \equiv 0$ is called a *measure of maximal entropy.*

Lemma 7.1.3. *[Sambarino [36], Lemma 2.4] If ϕ is a Hölder continuous flow on a compact metric space $\mathcal{X}$ and $f : \mathcal{X} \to \mathbb{R}$ is a positive Hölder continuous function, then*

$$\wp(\phi, -hf) = 0$$

if and only if $h = h_{top}(\phi^f)$. Moreover, if $h = h_{top}(\phi^f)$ and $\mathfrak{m}$ is an equilibrium state of $-hf$, then $\widehat{f.\mathfrak{m}}$ is a measure of maximal entropy for the reparameterized flow ϕ^f.

Theorem 7.1.4. *[Livšic] Let $f : \mathcal{X} \to \mathbb{R}$ be a Hölder continuous function, then $\langle \delta_a \mid f \rangle = 0$ for all $a \in \mathsf{O}$ if and only if f is Livšic cohomologous to zero.*

7.2 Entropy and pressure for Anosov flows

Let $f : \mathcal{X} \to \mathbb{R}_+$ be a positive Hölder continuous function and let T be a real number. We define

$$R_T(f) := \{a \in \mathsf{O} \mid \langle \delta_a \mid f \rangle \leqslant T\}.$$

We note that $R_T(f)$ only depends on the Livšic cohomology class of f.

Proposition 7.2.1. *[Bowen] The topological entropy of a topologically transitive metric Anosov flow $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ on a compact metric space $\mathcal{X}$ is finite and positive. Moreover,*

$$h_{top}(\phi) = \lim_{T\to\infty} \frac{1}{T} \log\left(\#\{a \in \mathsf{O} \mid p(a) \leqslant T\}\right).$$

In particular, for a nowhere vanishing Hölder continuous function f,

$$h_f := \lim_{T\to\infty} \frac{1}{T} \log\left(\#R_T(f)\right) = h_{top}(\phi^f)$$

is finite and positive.

Theorem 7.2.2. *[Bowen–Ruelle, Pollicott] Let $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ be a topologically transitive metric Anosov flow on a compact metric space $\mathcal{X}$ and let $g : \mathcal{X} \to \mathbb{R}$ be a Hölder continuous function, then there exists a unique equilibrium state $\mathfrak{m}_g$ for g. Moreover, if $f : \mathcal{X} \to \mathbb{R}$ is a Hölder continuous function such that $\mathfrak{m}_f = \mathfrak{m}_g$, then $f - g$ is Livšic cohomologous to a constant.*

We note that in such a situation it follows from [6] that the pressure function can be described in the following alternative way:

$$\wp(\phi, g) = \lim_{T\to\infty} \frac{1}{T} \log\left(\sum_{a\in R_T(1)} e^{\langle\delta_a|g\rangle}\right).$$

Theorem 7.2.3. *[Bowen] A topologically transitive metric Anosov flow ϕ on a compact metric space $\mathcal{X}$ has a unique probability measure $\mathfrak{m}_\phi$ of maximal entropy. Moreover,*

$$\mathfrak{m}_\phi = \lim_{T\to\infty}\left(\frac{1}{\#R_T(1)} \sum_{a\in R_T(1)} \widehat{\delta}_a\right).$$

The probability measure of maximal entropy for ϕ is called the *Bowen–Margulis measure* of ϕ.

7.3 Intersection and renormalised intersection

7.3.1 Intersection

Let ϕ be a topologically transitive metric Anosov flow on a compact metric space $\mathcal{X}$. Also let $f : \mathcal{X} \to \mathbb{R}_+$ be a positive Hölder continuous function and $g : \mathcal{X} \to \mathbb{R}$ be any continuous function.

Definition 7.3.1. *We define the intersection of f and g as follows:*

$$I(f, g) := \int \frac{g}{f} \mathsf{d}\mathfrak{m}_{\phi^f},$$

where $\mathfrak{m}_{\phi^f}$ is the Bowen-Margulis measure of the flow ϕ^f.

Using theorem 7.2.3 and equation 7.1.1 we get that

$$I(f,g) = \lim_{T\to\infty}\left(\frac{1}{\#R_T(f)}\sum_{a\in R_T(f)}\frac{\langle\delta_a \mid g\rangle}{\langle\delta_a \mid f\rangle}\right). \tag{7.3.1}$$

Morever, using the second part of the lemma 7.1.3 we have

$$I(f,g) = \frac{\int g \mathrm{d}\mathfrak{m}_{-h_f f}}{\int f \mathrm{d}\mathfrak{m}_{-h_f f}}$$

where h_f is the topological entropy of ϕ^f and $\mathfrak{m}_{-h_f.f}$ is the equilibrium state of $-h_f.f$. Now as $\langle\delta_a \mid f\rangle$ depends only on the Livšic cohomology class of f and $\langle\delta_a \mid g\rangle$ depends only on the Livšic cohomology class of g we get that the intersection $I(f,g)$ depends only on the Livšic cohomology classes of f and g.

7.3.2 Renormalized intersection

Definition 7.3.2. *Let $f,g : \mathcal{X} \to \mathbb{R}_+$ be two positive Hölder continuous functions. We define the renormalized intersection as follows:*

$$J(f,g) := \frac{h_g}{h_f} I(f,g),$$

where h_f and h_g are the topological entropies of ϕ^f and ϕ^g.

We note that the renormalized intersection J is not necessarily symmetric. Now using the uniqueness of the equilibrium states and the definition of the pressure we get that:

Proposition 7.3.3. *If ϕ is a topologically transitive metric Anosov flow on a compact metric space $\mathcal{X}$, and $f : \mathcal{X} \to \mathbb{R}_+$ and $g : \mathcal{X} \to \mathbb{R}_+$ are positive Hölder continuous functions, then*

$$J(f,g) \geqslant 1.$$

Moreover, $J(f,g) = 1$ if and only if $h_f f$ and $h_g g$ are Livšic cohomologous.

7.4 Variation of the pressure and the pressure form

We note that a more detailed version of the following constructions can be found in [8]. It is also similar to a construction that was introduced by McMullen in [29].

7.4.1 First and second derivatives

Let g be a Hölder continuous function. We say that g has *mean zero* with respect to f if and only if

$$\int g \mathrm{d}\mathfrak{m}_f = 0.$$

The *variance* of a mean zero Hölder continuous function g with respect to f is defined as follows:

$$\mathrm{Var}(g,\mathfrak{m}_f) := \lim_{T\to\infty}\frac{1}{T}\int\left(\int_0^T g(\phi_s x)ds\right)^2 \mathsf{d}\mathfrak{m}_f(x),$$

where $\mathfrak{m}_f$ is the equilibrium state of f. Similarly, for two mean zero Hölder continuous functions g and h, the *covariance* of g and h with respect to f is defined as follows:

$$\mathrm{Cov}(g,h,\mathfrak{m}_f) := \lim_{T\to\infty}\frac{1}{T}\int\left(\int_0^T g(\phi_s x)ds\right)\left(\int_0^T h(\phi_s x)ds\right)\mathsf{d}\mathfrak{m}_f(x).$$

Proposition 7.4.1. *[Parry–Pollicott, Ruelle] Suppose that ϕ is a topologically transitive metric Anosov flow on a compact metric space $\mathcal{X}$, and $f:\mathcal{X}\to\mathbb{R}$ and $g:\mathcal{X}\to\mathbb{R}$ are Hölder continuous functions. If $\mathfrak{m}_f$ is the equilibrium state of f, then*

1. *The function $t\to\wp(f+tg)$ is analytic,*
2. *The first derivative is given by*
$$\left.\frac{\partial}{\partial t}\right|_{t=0}\wp(\phi,f+tg) = \int g\mathsf{d}\mathfrak{m}_f,$$
3. *If $\int g\mathsf{d}\mathfrak{m}_f = 0$ then*
$$\left.\frac{\partial^2}{\partial t^2}\right|_{t=0}\wp(\phi,f+tg) = \mathit{Var}(g,\mathfrak{m}_f),$$
4. *If $\mathit{Var}(g,\mathfrak{m}_f)=0$ then g is Liv̆sic cohomologous to zero.*

7.4.2 The pressure form

Let $C^h(\mathcal{X})$ be the set of all real valued Hölder continuous functions on $\mathcal{X}$ and let $\mathcal{P}(\mathcal{X})$ be the set of all pressure zero Hölder continuous functions on $\mathcal{X}$, that is,

$$\mathcal{P}(\mathcal{X}) := \{\mathtt{f}\in C^h(\mathcal{X}) \mid \wp(\mathtt{f})=0\}.$$

The tangent space of $\mathcal{P}(\mathcal{X})$ at $\mathtt{f}$ is the set

$$\mathsf{T}_{\mathtt{f}}\mathcal{P}(\mathcal{X}) = \ker(\mathsf{d}_{\mathtt{f}}\wp) = \left\{g\in C^h(\mathcal{X}) \mid \int g\mathsf{d}\mathfrak{m}_{\mathtt{f}} = 0\right\}$$

where $\mathfrak{m}_{\mathtt{f}}$ is the equilibrium state of $\mathtt{f}$.

Definition 7.4.2. *The pressure semi-norm of $g\in\mathsf{T}_{\mathtt{f}}\mathcal{P}(\mathcal{X})$ is defined as follows:*

$$\|g\|_\wp^2 := -\frac{\mathit{Var}(g,\mathfrak{m}_{\mathtt{f}})}{\int \mathtt{f}\mathsf{d}\mathfrak{m}_{\mathtt{f}}}.$$

Lemma 7.4.3. *Let ϕ be a topologically transitive metric Anosov flow on a compact metric space $\mathcal{X}$. If $\{\mathtt{f}_t\}_{t\in(-1,1)}$ is a smooth one-parameter family contained in $\mathcal{P}(\mathcal{X})$ then*

$$\|\dot{\mathtt{f}}_0\|_\wp^2 = \frac{\int\ddot{\mathtt{f}}_0\mathsf{d}\mathfrak{m}_{\mathtt{f}_0}}{\int\mathtt{f}_0\mathsf{d}\mathfrak{m}_{\mathtt{f}_0}}.$$

We note that the following result can also be found in [8] and is a generalized version of a previous work done by Bonahon in [4]:

Proposition 7.4.4. *Let ϕ be a topologically transitive metric Anosov flow on a compact metric space $\mathcal{X}$. If*

$$\{f_t : \mathcal{X} \to \mathbb{R}_+\}_{t\in(-1,1)}$$

is a one-parameter family of positive Hölder continuous functions and $\mathtt{f}_t = -h_{f_t} f_t$ for all $t \in (-1,1)$, then

$$\left.\frac{\partial^2}{\partial t^2}\right|_{t=0} J(f_0, f_t) = \|\dot{\mathtt{f}}_0\|^2_{\wp}.$$

Therefore the pressure semi-norm arises naturally from the *pressure form* $\mathtt{P}$ which is the symmetric 2-tensor on $\mathsf{T}_{\mathtt{f}}\mathcal{P}(\mathcal{X})$ given by the Hessian of $J_{\mathtt{f}} := J(\mathtt{f}, .)$. We also note that if $f, g \in \mathsf{T}_{\mathtt{f}}\mathcal{P}(\mathcal{X})$, then

$$\mathtt{P}(f,g) = -\frac{\mathrm{Cov}(f, g, \mathfrak{m}_{\mathtt{f}})}{\int \mathtt{f} d\mathfrak{m}_{\mathtt{f}}}.$$

Proposition 7.4.5. *Let ϕ be a topologically transitive metric Anosov flow on a compact metric space $\mathcal{X}$. Let $\{f_u : \mathcal{X} \to \mathbb{R}\}_{u\in D}$ and $\{g_v : \mathcal{X} \to \mathbb{R}\}_{v\in D}$ be two analytic families of Hölder continuous functions. Then the function*

$$u \mapsto \wp(\phi, f_u)$$

is analytic. Moreover, if the family $\{f_u\}_{u\in D}$ consists of positive functions then the functions

$$u \mapsto h_{f_u}$$

and

$$(u,v) \mapsto I(f_u, g_v)$$

are both analytic.

Chapter 8

Deformation Theory

8.1 Transverse Analyticity

In this section we mention some definitions and theorems introduced by Hirsch–Pugh–Shub in [22] and which appeared in more details in [8]. We use these theorems to prove the analyticity results in coming sections.

Definition 8.1.1. *[Transversely regular functions] Let $\mathcal{D}^{\mathbb{C}}$ be a complex disk, let $\mathcal{X}$ be a compact metric space and let M be a complex analytic manifold. A continuous function*

$$f : \mathcal{D}^{\mathbb{C}} \times \mathcal{X} \to M$$

is called transversely complex analytic if and only if the following two conditions are satisfied:

1. *The function*

$$\begin{aligned} f_x : \mathcal{D}^{\mathbb{C}} &\longrightarrow M \\ u &\longmapsto f(u, x) \end{aligned}$$

is complex analytic for every $x \in \mathcal{X}$.

2. *The function from $\mathcal{X}$ to $\mathcal{C}^{\omega}(\mathcal{D}^{\mathbb{C}}, M)$ given by $x \mapsto f_x$ is continuous.*

Furthermore, the function f is called μ-Hölder (or Lipschitz) transversely complex analytic if and only if the map in (2) is μ-Hölder (or Lipschitz) continuous.

Similarly, μ-Hölder (or Lipschitz) *transversely real analytic* functions can be defined by replacing the complex disk $\mathcal{D}^{\mathbb{C}}$ by a real disk $\mathcal{D}$, replacing the complex analytic manifold M by a real analytic manifold and by requiring that the maps in (1) are real analytic and requiring in (2) that the map from $\mathcal{X}$ to $\mathcal{C}^{\omega}(\mathcal{D}, M)$ is μ-Hölder (or Lipschitz).

In a similar fashion transverse regularity of bundles is defined in terms of the transverse regularity of their trivializations.

Definition 8.1.2. *[Transversely regular bundles] Let M be a complex analytic manifold and let*

$$\pi : \mathbb{E} \to \mathcal{D}^{\mathbb{C}} \times \mathcal{X}$$

be a bundle whose fibers are M. The bundle $\pi : \mathbb{E} \to \mathcal{D}^{\mathbb{C}} \times \mathcal{X}$ is called transversely complex analytic if and only if it admits a family of trivializations of the form $\{\mathcal{D}^{\mathbb{C}} \times \mathcal{U}_\beta \times M\}$ (where the collection $\{\mathcal{U}_\beta\}$ is an open cover of $\mathcal{X}$) so that the the corresponding change of coordinate functions are transversely complex analytic. Similarly, the bundle $\pi : \mathbb{E} \to \mathcal{D}^{\mathbb{C}} \times \mathcal{X}$ is called μ-Hölder (or Lipschitz) transversely complex analytic if and only if it admits a family of trivializations so that the the corresponding change of coordinate functions are μ-Hölder (or Lipschitz) transversely complex analytic.

In such a case, a section σ of $\mathbb{E}$ is called μ-Hölder (or Lipschitz) transversely complex analytic, if and only if in any of the trivializations the corresponding map to M is μ-Hölder (or Lipschitz) transversely complex analytic.

Similarly, μ-Hölder (or Lipschitz) transversely real analytic bundles and sections can be defined by replacing the complex disk $\mathcal{D}^{\mathbb{C}}$ with a real disk $\mathcal{D}$ and the complex analytic manifold M with a real analytic manifold.

Theorem 8.1.3. *Let $\mathcal{X}$ be a compact metric space and let M be a complex analytic manifold. Suppose that $\pi : E \to D \times \mathcal{X}$ is a Lipschitz transversely complex analytic bundle with fibre M and D is a complex (or real) disk. Let $f : \mathcal{X} \to \mathcal{X}$ be a Lipschitz homeomorphism and let F be a Lipschitz transversely complex analytic bundle automorphism of E lifting $\mathsf{id} \times f$. Suppose that σ_0 is a section of the restriction of E over $\{0\} \times \mathcal{X}$ which is fixed by F and that F contracts along σ_0. Then there exists a neighborhood U containing 0 in D, a positive real number $\mu > 0$, a μ-Hölder transversely complex analytic section η over $U \times \mathcal{X}$ and a neighborhood B of $\eta(U \times \mathcal{X})$ in $\pi^{-1}(U \times \mathcal{X})$ such that*

1. *The bundle automorphism F fixes η,*
2. *The bundle automorphism F contracts E along η,*
3. *The restriction $\eta|_{\{0\}\times\mathcal{X}} = \sigma_0$, and*
4. *If $\zeta : U \times \mathcal{X} \to E$ is a section with $\zeta(U \times \mathcal{X}) \subset B$ and ζ is fixed by F, then $\zeta = \eta$.*

Definition 8.1.4. *Let U be a subset of D. We say that a section σ over $U \times \mathcal{X}$ is fixed by F if and only if*

$$F(\sigma(u, x)) = \sigma(u, f(x)).$$

In such a case, we further say that F contracts along σ if there exists a continuously varying fibrewise Riemannian metric $\|.\|$ on the bundle E such that if

$$\mathsf{D}^f F_{\sigma(u,x)} : \mathsf{T}_{\sigma(u,x)}\pi^{-1}(u, x) \to \mathsf{T}_{\sigma(u,f(x))}\pi^{-1}(u, f(x))$$

is the fibrewise tangent map, then

$$\|\mathsf{D}^f F_{\sigma(u,x)}\| < 1.$$

The following result has been taken from [8]. A similar statement appeared in Hubbard [23].

Lemma 8.1.5. *Let D be a complex (or real) disk, let M be a complex analytic manifold, let $\mathcal{X}$ be a compact metric space and let $f : D \times \mathcal{X} \to M$ be a μ-Hölder transversely complex analytic function, then the map*

$$\begin{aligned} \hat{f} : D &\to C^{\mu}(\mathcal{X}, M) \\ u &\mapsto f_u \end{aligned}$$

is complex analytic, where $f_u(.) := f(u, .)$.

8.2 Analyticity of limit maps

In this section we show that the limit maps vary analytically over the analytic manifold $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$. The proofs given in this section are inspired by some of the proofs given in the section 6 of [8].

Theorem 8.2.1. *Let $\{\rho_u\}_{u\in\mathcal{D}}$ be a real analytic family in $\mathsf{Hom}(\Gamma, \mathsf{G})$ parameterized by a disk $\mathcal{D}$ around 0. If ρ_0 is $(\mathsf{G}, \mathsf{P}^{\pm})$-Anosov with limit maps*

$$\xi_0^{\pm} : \partial_\infty\Gamma \longrightarrow \mathsf{G}/\mathsf{P}^{\pm}$$

then there exists a sub-disk $\mathcal{D}_0$ of $\mathcal{D}$ (containing 0), a positive real number μ and a continuous map

$$\xi^{+} : \mathcal{D}_0 \times \partial_\infty\Gamma \longrightarrow \mathsf{G}/\mathsf{P}^{+}$$

with the following properties:

1. *If u is in $\mathcal{D}_0$ then ρ_u is a $(\mathsf{G}, \mathsf{P}^{\pm})$-Anosov representation with μ-Hölder limit map given by*

$$\begin{aligned} \xi_u^{+} : \partial_\infty\Gamma &\longrightarrow \mathsf{G}/\mathsf{P}^{+} \\ x &\longmapsto \xi^{+}(u, x), \end{aligned}$$

2. *If x is in $\partial_\infty\Gamma$ then the following map is real analytic*

$$\begin{aligned} \xi_x^{+} : \mathcal{D}_0 &\longrightarrow \mathsf{G}/\mathsf{P}^{+} \\ u &\longmapsto \xi^{+}(u, x), \end{aligned}$$

3. *The map from $\partial_\infty\Gamma$ to $\mathcal{C}^{\omega}(\mathcal{D}_0, \mathsf{G}/\mathsf{P}^{+})$ given by $x \mapsto \xi_x^{+}$ is μ-Hölder,*

4. *The map from $\mathcal{D}_0$ to $\mathcal{C}^{\mu}(\partial_\infty\Gamma, \mathsf{G}/\mathsf{P}^{+})$ given by $u \mapsto \xi_u^{+}$ is real analytic.*

We will prove Theorem 8.2.1 using the following more general result.

Theorem 8.2.2. *Let $\{\rho_u\}_{u\in\mathcal{D}^{\mathbb{C}}}$ be a complex analytic family in $\mathsf{Hom}(\Gamma, \mathsf{G}_{\mathbb{C}})$ parameterized by a disk $\mathcal{D}^{\mathbb{C}}$ around 0. If ρ_0 is $(\mathsf{G}_{\mathbb{C}}, \mathsf{P}_{\mathbb{C}}^{\pm})$-Anosov with limit maps*

$$\xi_0^{\pm} : \partial_\infty\Gamma \to \mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^{\pm}$$

then there exists a sub-disk $\mathcal{D}_0^{\mathbb{C}}$ of $\mathcal{D}^{\mathbb{C}}$ (containing 0), a positive real number μ and a continuous map

$$\xi^{+} : \mathcal{D}_0^{\mathbb{C}} \times \partial_\infty\Gamma \to \mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^{+}$$

with the following properties:

1. *If u is in $\mathcal{D}_0^{\mathbb{C}}$ then ρ_u is a $(\mathsf{G}_{\mathbb{C}}, \mathsf{P}_{\mathbb{C}}^{\pm})$-Anosov representation with μ-Hölder limit map given by*

$$\begin{aligned} \xi_u^+ : \partial_\infty \Gamma &\longrightarrow \mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^+ \\ x &\longmapsto \xi^+(u, x), \end{aligned}$$

2. *If x is in $\partial_\infty \Gamma$ then the following map is complex analytic*

$$\begin{aligned} \xi_x^+ : \mathcal{D}_0^{\mathbb{C}} &\longrightarrow \mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^+ \\ u &\longmapsto \xi^+(u, x), \end{aligned}$$

3. *The map from $\partial_\infty \Gamma$ to $\mathcal{C}^\omega(\mathcal{D}_0^{\mathbb{C}}, \mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^+)$ given by $x \mapsto \xi_x^+$ is μ-Hölder,*

4. *The map from $\mathcal{D}_0^{\mathbb{C}}$ to $\mathcal{C}^\mu(\partial_\infty \Gamma, \mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^+)$ given by $u \mapsto \xi_u^+$ is complex analytic.*

Proof. Let $\{\rho_u\}_{u \in \mathcal{D}^{\mathbb{C}}} \subset \mathsf{Hom}(\Gamma, \mathsf{G}_{\mathbb{C}})$ be a complex analytic family of homomorphisms such that ρ_0 is $(\mathsf{G}_{\mathbb{C}}, \mathsf{P}_{\mathbb{C}}^{\pm})$-Anosov. Let us consider the trivial $\mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^+$-bundle over $\mathcal{D}^{\mathbb{C}} \times \widetilde{\mathsf{U}_0\Gamma}$ as follows:

$$\pi : \tilde{A} := \mathcal{D}^{\mathbb{C}} \times \widetilde{\mathsf{U}_0\Gamma} \times \mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^+ \longrightarrow \mathcal{D}^{\mathbb{C}} \times \widetilde{\mathsf{U}_0\Gamma}.$$

Furthermore, we consider the following action of Γ on $\tilde{A}$

$$\gamma(u, x, [g]) = (u, \gamma(x), [\rho_u(\gamma) g])$$

where γ is in Γ and notice that the quotient bundle $A := \Gamma \backslash \tilde{A}$ is a Lipschitz transversely complex analytic $\mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^+$-bundle over $\mathcal{D}^{\mathbb{C}} \times \mathsf{U}_0\Gamma$. The geodesic flows $\{\tilde{\psi}_t\}_{t \in \mathbb{R}}$ and $\{\psi_t\}_{t \in \mathbb{R}}$ respectively on $\widetilde{\mathsf{U}_0\Gamma}$ and $\mathsf{U}_0\Gamma$ lift to geodesic flows $\{\tilde{\Psi}_t\}_{t \in \mathbb{R}}$ and $\{\Psi_t\}_{t \in \mathbb{R}}$ on $\tilde{A}$ and A respectively. We note that the flow $\{\tilde{\Psi}_t\}_{t \in \mathbb{R}}$ acts trivially on the $\mathcal{D}^{\mathbb{C}}$ and $\mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^+$ factors. Now as ρ_0 is $(\mathsf{G}_{\mathbb{C}}, \mathsf{P}_{\mathbb{C}}^{\pm})$-Anosov with limit maps

$$\xi_0^{\pm} : \partial_\infty \Gamma \to \mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^{\pm},$$

the following map $\tilde{\sigma}_0$ defines a Γ-equivariant section of the restriction of the bundle $\tilde{A}$ over $\{0\} \times \widetilde{\mathsf{U}_0\Gamma}$,

$$\begin{aligned} \tilde{\sigma}_0 : \{0\} \times \widetilde{\mathsf{U}_0\Gamma} &\longrightarrow \tilde{A} \\ (0, (x, y, t)) &\longmapsto (0, (x, y, t), \xi_0^+(x)). \end{aligned}$$

Therefore the section $\tilde{\sigma}_0$ gives rise to a section σ_0 of A over $\{0\} \times \mathsf{U}_0\Gamma$.

Since ρ_0 is $(\mathsf{G}_{\mathbb{C}}, \mathsf{P}_{\mathbb{C}}^{\pm})$-Anosov, the bundle $\Xi_{\rho_0}^+$ over $\{0\} \times \mathsf{U}_0\Gamma$ with fiber $\mathsf{T}_{\sigma_0(0,\mathfrak{X})} \pi^{-1}(0, \mathfrak{X})$ gets contracted by the lift of the geodesic flow ψ_t as t goes to ∞. Hence there exists a real number t_0 such that for all $\mathfrak{X}$ in $\mathsf{U}_0\Gamma$ we have

$$\left\| \left(\mathsf{D}^{\psi_{t_0}} \Psi_{t_0} \right)_{\sigma_0(0,\mathfrak{X})} \right\| < 1$$

where

$$\left(\mathsf{D}^{\psi_{t_0}}\Psi_{t_0}\right)_{\sigma_0(0,\mathfrak{X})} : \mathsf{T}_{\sigma_0(0,\mathfrak{X})}\pi^{-1}(0,\mathfrak{X}) \to \mathsf{T}_{\sigma_0(0,\psi_{t_0}\mathfrak{X})}\pi^{-1}(0,\psi_{t_0}\mathfrak{X})$$

is the fiberwise map of the bundle automorphism induced by ψ_{t_0} or in short "lift of ψ_{t_0}". Now using theorem 8.1.3 we get that there exists a sub-disk $\mathcal{D}_1^{\mathbb{C}} \subset \mathcal{D}^{\mathbb{C}}$ containing 0, a positive real number μ, and a μ-Hölder transversely complex analytic section

$$\sigma : \mathcal{D}_1^{\mathbb{C}} \times \mathsf{U}_0\Gamma \to A$$

that extends σ_0, is fixed by Ψ_{t_0} and such that for all $\mathfrak{X}$ in $\mathsf{U}_0\Gamma$ and u in $\mathcal{D}_1^{\mathbb{C}}$ we have

$$\left\|\left(\mathsf{D}^{\psi_{t_0}}\Psi_{t_0}\right)_{\sigma(u,\mathfrak{X})}\right\| < 1.$$

We now use the uniqueness portion of the theorem 8.1.3 to deduce that σ is fixed by Ψ_t for all real number t. Therefore we get that there exists a sub-disk $\mathcal{D}_1^{\mathbb{C}} \subset \mathcal{D}^{\mathbb{C}}$ containing 0, a positive real number μ, and a μ-Hölder transversely complex analytic section σ of the bundle A that extends σ_0, is fixed by the flow $\{\Psi_t\}_{t\in\mathbb{R}}$ and such that Ψ_t is contracting along σ as t goes to ∞. Now we can lift the section σ to get a section $\tilde{\sigma}$ as follows:

$$\tilde{\sigma} : \mathcal{D}_1^{\mathbb{C}} \times \widetilde{\mathsf{U}_0\Gamma} \to \tilde{A} = \mathcal{D}_1^{\mathbb{C}} \times \widetilde{\mathsf{U}_0\Gamma} \times \mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^{+}.$$

Let π_3 be the projection of $\mathcal{D}_1^{\mathbb{C}} \times \widetilde{\mathsf{U}_0\Gamma} \times \mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^{+}$ onto $\mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^{+}$. Therefore we get a map

$$\eta := \pi_3 \circ \tilde{\sigma} : \mathcal{D}_1^{\mathbb{C}} \times \widetilde{\mathsf{U}_0\Gamma} \to \mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^{+}.$$

Since $\tilde{\sigma}$ is fixed by the flow $\{\Psi_t\}_{t\in\mathbb{R}}$ we get that the map η is invariant under the flow $\{\psi_t\}_{t\in\mathbb{R}}$. Hence $\eta(u,(x,y,t))$ is independent of the variable t.
Now let γ be an infinite order element of Γ with period t_γ. We notice that as $\eta_u(\gamma^-,\gamma^+,0)$ is independent of the variable t we have

$$\gamma^{-n}\eta_u(\gamma^-,\gamma^+,0) = \eta_u(\gamma^-,\gamma^+,-nt_\gamma) = \eta_u(\gamma^-,\gamma^+,0)$$

and hence $\eta_u(\gamma^-,\gamma^+,0)$ is a fixed point of γ^{-1}. We claim that it is an attracting fixed point. Indeed, as $\tilde{\Psi}_t$ is contracting as t goes to ∞ and as $\|.\|$ is Γ-equivariant we have for all X in $\mathsf{T}_{\eta_u(\gamma^-,\gamma^+,0)}\mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^{+}$ that

$$\begin{aligned}
\|\gamma^{-n}X\|_{\eta_u(\gamma^-,\gamma^+,0)} &= \|X\|_{\eta_u(\gamma^n(\gamma^-,\gamma^+,0))}\\
&= \|X\|_{\eta_u(\gamma^-,\gamma^+,nt_\gamma)}\\
&= \|X\|_{\tilde{\Psi}_{nt_\gamma}\eta_u(\gamma^-,\gamma^+,0)} \leqslant Ae^{-ct_\gamma n}\|X\|_{\eta_u(\gamma^-,\gamma^+,0)}.
\end{aligned}$$

Hence for m large enough the operator norm $\|\gamma^{-m}\| < 1$ and we have that there exists a ball $\mathsf{B}_d(\eta_u(\gamma^-,\gamma^+,0),k_0)$ of radius k_0 around $\eta_u(\gamma^-,\gamma^+,0)$ for some metric d on $\mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^{+}$ such that γ^{-m} is contracting on the ball. Hence γ^{-1} is also contracting on the ball. We call the ball $\mathsf{B}_d(\eta_u(\gamma^-,\gamma^+,0),k_0)$ a *basin of convergence* for the action of γ^{-1} around $\eta_u(\gamma^-,\gamma^+,0)$. Therefore in particular for any sequence $\{p_n\}_{n\in\mathbb{N}}$ in $\mathsf{B}_d(\eta_u(\gamma^-,\gamma^+,0),k_0)$ we have that

$$\lim_{n\to\infty} d(\eta_u(\gamma^-,\gamma^+,0),\gamma^{-n}p_n) = 0.$$

Moreover, for any Γ-invariant metric $\mathfrak{d}$ on $\widetilde{\mathsf{U}_0\Gamma}$ and given any $z \in \partial_\infty\Gamma$ there exist t_z such that

$$\lim_{t\to-\infty} \mathfrak{d}(\tilde{\psi}_t(\gamma^-, \gamma^+, 0), \tilde{\psi}_t(\gamma^-, z, t_z)) = 0.$$

Hence

$$\begin{aligned} 0 &= \lim_{n\to\infty} \mathfrak{d}((\gamma^-, \gamma^+, -nt_\gamma), (\gamma^-, z, t_z - nt_\gamma)) \\ &= \lim_{n\to\infty} \mathfrak{d}(\gamma^{-n}(\gamma^-, \gamma^+, 0), (\gamma^-, z, t_z - nt_\gamma)) \\ &= \lim_{n\to\infty} \mathfrak{d}((\gamma^-, \gamma^+, 0), \gamma^n(\gamma^-, z, t_z - nt_\gamma)). \end{aligned}$$

Therefore if we take

$$p_n = \eta_u(\gamma^n(\gamma^-, z, t_z - nt_\gamma))$$

then the sequence is eventually in $\mathsf{B}_d(\eta_u(\gamma^-, \gamma^+, 0), k_0)$ and we get that

$$\begin{aligned} 0 &= \lim_{n\to\infty} d(\eta_u(\gamma^-, \gamma^+, 0), \gamma^{-n}\eta_u(\gamma^n(\gamma^-, z, t_z - nt_\gamma))) \\ &= \lim_{n\to\infty} d(\eta_u(\gamma^-, \gamma^+, 0), \eta_u(\gamma^-, z, t_z - nt_\gamma)). \end{aligned}$$

Now as $\eta(u, (x, y, t))$ is independent of t we get that

$$0 = \lim_{n\to\infty} d(\eta_u(\gamma^-, \gamma^+, 0), \eta_u(\gamma^-, z, 0))$$

and hence $\eta_u(\gamma^-, \gamma^+, 0) = \eta_u(\gamma^-, z, 0)$. Moreover, as the fixed points of infinite order elements are dense in $\partial_\infty\Gamma$ we conclude that $\eta(u, (x, y, t))$ is independent of the variable y. Therefore there exists a Γ-equivariant Hölder transversely complex analytic map

$$\xi^+ : \mathcal{D}_1^{\mathbb{C}} \times \partial_\infty\Gamma \to \mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^+$$

extending the map ξ_0^+. In a similar way we get that there exists a sub-disk $\mathcal{D}_2^{\mathbb{C}} \subset \mathcal{D}^{\mathbb{C}}$ containing 0 such that there exists a Γ-equivariant Hölder transversely complex analytic map

$$\xi^- : \mathcal{D}_1^{\mathbb{C}} \times \partial_\infty\Gamma \to \mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^-$$

extending the map ξ_0^-.
Moreover, we recall that $\mathcal{N}_{\mathbb{C}}$ is open in $\mathbb{X}_{\mathbb{C}} \times \mathbb{X}_{\mathbb{C}}$ and we know that

$$\mathsf{G}_{\mathbb{C}}/\mathsf{L}_{\mathbb{C}} \cong \mathcal{N}_{\mathbb{C}}.$$

Hence $\mathsf{G}_{\mathbb{C}}/\mathsf{L}_{\mathbb{C}}$ is an open subset of $\mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^+ \times \mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}}^-$. Now as

$$(\xi_0^+, \xi_0^-)(\{0\} \times \partial_\infty\Gamma^{(2)}) \subset \mathsf{G}_{\mathbb{C}}/\mathsf{L}_{\mathbb{C}},$$

we get that there exists a sub-disk $\mathcal{D}_0^{\mathbb{C}} \subset \mathcal{D}_1^{\mathbb{C}} \cap \mathcal{D}_2^{\mathbb{C}}$ containing 0 such that $(\xi^+, \xi^-)(\mathcal{D}_0^{\mathbb{C}} \times \partial_\infty\Gamma^{(2)}) \subset \mathsf{G}_{\mathbb{C}}/\mathsf{L}_{\mathbb{C}}$. Therefore we have proved properties (1), (2) and (3). Now property (4) follows from lemma 8.1.5 and this completes the proof of Theorem 8.2.2. $\square$

We now show that Theorem 8.2.1 follows from Theorem 8.2.2.

Proof. Let $\{\rho_u\}_{u\in\mathcal{D}} \subset \mathsf{Hom}(\Gamma, \mathsf{G})$ be a real analytic family of homomorphisms such that ρ_0 is $(\mathsf{G}, \mathsf{P}^{\pm})$-Anosov. We observe that a $(\mathsf{G}, \mathsf{P}^{\pm})$-Anosov representation is also a $(\mathsf{G}_{\mathbb{C}}, \mathsf{P}^{\pm}_{\mathbb{C}})$-Anosov representation. Now on a sub-disk $\mathcal{D}_3$ of $\mathcal{D}$, containing 0, we can extend $\{\rho_u\}_{u\in\mathcal{D}_3}$ to a complex analytic family of representations $\{\rho_u\}_{u\in\mathcal{D}^{\mathbb{C}}_3} \subset \mathsf{Hom}(\Gamma, \mathsf{G}_{\mathbb{C}})$, where $\mathcal{D}^{\mathbb{C}}_3$ is the complexification of $\mathcal{D}_3$. Now using theorem 8.2.2 we get that there exists a Γ-equivariant Hölder transversely complex analytic map

$$\xi^+ : \mathcal{D}^{\mathbb{C}}_0 \times \partial_\infty\Gamma \to \mathsf{G}_{\mathbb{C}}/\mathsf{P}^+_{\mathbb{C}}$$

extending the limit map

$$\xi^+_0 : \{0\} \times \partial_\infty\Gamma \to \mathsf{G}_{\mathbb{C}}/\mathsf{P}^+_{\mathbb{C}}.$$

We claim that there exist a sub-disk $\mathcal{D}_{01} \subset \mathcal{D}_0$, containing 0, such that

$$\xi^+(\mathcal{D}_{01} \times \partial_\infty\Gamma) \subset \mathsf{G}/\mathsf{P}^+.$$

Indeed, to begin with we notice that $\xi^+(\{0\} \times \partial_\infty\Gamma) \subset \mathsf{G}/\mathsf{P}^+$. Now using theorem 8.1.3 (4) we get that there exist a sub-disk $\mathcal{D}^{\mathbb{C}}_4 \subset \mathcal{D}^{\mathbb{C}}_0$, containing 0, and a neighborhood B of $\xi^+(\mathcal{D}^{\mathbb{C}}_4 \times \partial_\infty\Gamma)$ such that the limit map is unique in B. Let $\mathfrak{i}$ be the anti-holomorphic involution on $\mathsf{G}_{\mathbb{C}}/\mathsf{P}^+_{\mathbb{C}}$. As $\mathfrak{i}$ is continuous and $\mathfrak{i} \circ \xi^+_0 = \xi^+_0$ we obtain that there exist a sub-disk $\mathcal{D}^{\mathbb{C}}_5 \subset \mathcal{D}^{\mathbb{C}}_0$, containing 0, such that

$$\mathfrak{i} \circ \xi^+(\mathcal{D}^{\mathbb{C}}_5 \times \partial_\infty\Gamma) \subset \mathsf{B}.$$

We define

$$\mathcal{D}^{\mathbb{C}}_{01} := \mathcal{D}^{\mathbb{C}}_4 \cap \mathcal{D}^{\mathbb{C}}_5$$

and by local uniqueness of the limit map we notice that for all u in $\mathcal{D}^{\mathbb{C}}_{01}$ the following holds:

$$\mathfrak{i} \circ \xi^+_u = \xi^+_{iu}.$$

Now for all u in $\mathcal{D}^{\mathbb{C}}_{01}$ satisfying $\mathfrak{i} \circ \rho_u = \rho_u$ we get that $u = iu$ and hence we conclude that

$$\mathfrak{i} \circ \xi^+_u = \xi^+_u.$$

We also note that the restrictions of complex analytic functions to real analytic submanifolds are real analytic. Therefore the map $\xi^+|_{\mathcal{D}_{01}}$ satisfies all the properties required by Theorem 8.2.1. $\square$

8.3 Analyticity of Reparametrizations

Let $\mathsf{U}_0\Gamma$ be the Gromov geodesic flow of the free group Γ and let ρ be an element of $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$. Moreover, let $\Sigma_{\mathsf{L}(\rho)} := \mathsf{L}_\rho(\Gamma)\backslash\mathbb{H}$ and $\mathsf{M}_\rho := \rho(\Gamma)\backslash\mathbb{A}$. Now as Γ is a free group we have an orbit equivalent homeomorphism between $\mathsf{U}_0\Gamma$ and $\mathsf{U}_{\mathrm{rec}}\Sigma_{\mathsf{L}(\rho)}$. Moreover, the flow on $\mathsf{U}_{\mathrm{rec}}\Sigma_{\mathsf{L}(\rho)}$ coming from the geodesic flow on $\mathsf{U}\Sigma_{\mathsf{L}(\rho)}$ is a Hölder reparametrization of the Gromov flow on $\mathsf{U}_0\Gamma$. Also from [18] and [17] we know that there exists an orbit equivalent homeomorphism between $\mathsf{U}_{\mathrm{rec}}\Sigma_{\mathsf{L}(\rho)}$ and $\mathsf{U}_{\mathrm{rec}}\mathsf{M}_\rho$ such that the flow on $\mathsf{U}_{\mathrm{rec}}\mathsf{M}_\rho$ coming from the affine linear flow is a Hölder reparametrization of the flow on $\mathsf{U}_{\mathrm{rec}}\Sigma_{\mathsf{L}(\rho)}$

coming from the geodesic flow on $\mathsf{U}\Sigma_{\mathsf{L}(\rho)}$. Therefore there exist an orbit equivalent homeomorphism between $\mathsf{U}_0\Gamma$ and $\mathsf{U}_{\mathrm{rec}}\mathsf{M}_\rho$ such that the affine linear flow on $\mathsf{U}_{\mathrm{rec}}\mathsf{M}_\rho$ is a Hölder reparametrization of the Gromov flow. Hence for any $\rho \in \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ we get a positive Hölder continuous map

$$\mathfrak{f}_\rho : \mathsf{U}_0\Gamma \to \mathbb{R}$$

which gives the reparametrization. We recall that positivity follows from lemma 3 of [17]. We further note that for all $\gamma \in \Gamma$ we have

$$\int_\gamma \mathfrak{f}_\rho = \alpha_\rho(\gamma).$$

Proposition 8.3.1. *Let $\{\rho_u\}_{u\in\mathcal{D}}$ be a real analytic family of homomorphisms ρ_u in $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ parameterized by a disk $\mathcal{D}$ around 0. Then there exists a sub-disk $\mathcal{D}_1$ around 0 and a real analytic family*

$$\{\mathsf{f}_u : \mathsf{U}_0\Gamma \to \mathbb{R}\}_{u\in\mathcal{D}_1}$$

of positive Hölder continuous functions such that the function f_u is Liv̆sic cohomologous to the function $\mathfrak{f}_{\rho_u}$.

Proof. We start by constructing the following line bundle:

$$\mathcal{B} := \left\{ \left((X, v), P_{X,v,v^+}, P_{X,v,v^-}\right) \mid (X, v) \in \mathsf{U}\mathbb{A} \right\} \tag{8.3.1}$$

is a line bundle over G/L. Now using proposition 6.4.2 and theorem 8.2.1 we get that there exist a sub-disk $\mathcal{D}_0 \subset \mathcal{D}$, containing 0, and μ-Hölder transversely real analytic maps,

$$(\xi^+, \xi^-) : \mathcal{D}_0 \times \partial_\infty\Gamma^{(2)} \to \mathsf{G}/\mathsf{L}. \tag{8.3.2}$$

Let us consider the the projection map,

$$\begin{aligned} \pi : \mathcal{D}_0 \times \widetilde{\mathsf{U}_0\Gamma} &\to \mathcal{D}_0 \times \partial_\infty\Gamma^{(2)} \\ (u, (x, y, t)) &\mapsto (u, (x, y)) \end{aligned} \tag{8.3.3}$$

and note that the map $(\xi^+, \xi^-) \circ \pi$ is μ-Hölder transversely real analytic. We take the pullback of this map to define a μ-Hölder transversely real analytic bundle $\tilde{\mathcal{B}} := ((\xi^+, \xi^-) \circ \pi)^* \mathcal{B}$ over $\mathcal{D}_0 \times \widetilde{\mathsf{U}_0\Gamma}$. The free group Γ acts on this bundle as follows:

$$\begin{aligned} &\gamma. \left(u, (x, y, t), \left((X, v), \xi_u^+(x, y, t), \xi_u^-(x, y, t)\right)\right) \\ &:= \left(u, \gamma.(x, y, t), \left(\left(\rho_u(\gamma)X, \mathsf{L}_{\rho_u(\gamma)}v\right), \xi_u^+(\gamma(x, y, t)), \xi_u^-(\gamma(x, y, t))\right)\right) \end{aligned}$$

We observe that the action of Γ gives rise to a quotient bundle $\Gamma\backslash\tilde{\mathcal{B}}$ over $\mathcal{D}_0 \times \mathsf{U}_0\Gamma$. Let σ be a μ-Hölder transversely real analytic section of this bundle and let $\tilde{\sigma}$ be its lift onto $\mathcal{D}_0 \times \widetilde{\mathsf{U}_0\Gamma}$. Let $\{\tilde{\psi}_t\}_{t\in\mathbb{R}}$ be the flow on $\mathcal{D}_0 \times \widetilde{\mathsf{U}_0\Gamma}$ such that $\tilde{\psi}_t(u, (x, y, t_0)) := (u, (x, y, t + t_0))$. Also let π_1, π_2 denote the map which sends $\left((X, v), P_{X,v,v^+}, P_{X,v,v^-}\right)$ to X and v respectively. We observe that for all real number t

$$\begin{aligned} \pi_1\tilde{\psi}_t^*\tilde{\sigma}(u, (x, y, t_0)) =& \pi_1\tilde{\sigma}(u, (x, y, t_0)) \\ &+ k_t(u, (x, y, t_0))\pi_2\tilde{\sigma}(u, (x, y, t_0)) \end{aligned} \tag{8.3.4}$$

where $k_t : \mathcal{D}_0 \times \widetilde{\mathsf{U}_0\Gamma} \to \mathbb{R}$ is a μ-Hölder transversely real analytic function and for all real number t

$$\pi_2 \tilde{\psi}_t^* \tilde{\sigma}(u, (x, y, t_0)) = \pi_2 \tilde{\sigma}(u, (x, y, t_0)). \tag{8.3.5}$$

Let t_γ be the period of the geodesic $\{(\gamma^-, \gamma^+, t) \mid t \in \mathbb{R}\}$ fixed by γ in Γ. We further notice that

$$\begin{aligned}
\mathtt{L}_{\rho_u}(\gamma) \pi_2 \tilde{\sigma}(u, (\gamma^-, \gamma^+, t_0)) &= \pi_2 \tilde{\sigma}(u, \gamma(\gamma^-, \gamma^+, t_0)) \\
&= \pi_2 \tilde{\sigma}(u, (\gamma^-, \gamma^+, t_0 + t_\gamma)) \\
&= \pi_2 \tilde{\sigma}(u, (\gamma^-, \gamma^+, t_0)).
\end{aligned}$$

We also recall that $\pi_2 \tilde{\sigma}(0, (\gamma^-, \gamma^+, t_0)) = \nu_{\rho_0}(\gamma^-, \gamma^+)$. Therefore we deduce that

$$\pi_2 \tilde{\sigma}(u, (\gamma^-, \gamma^+, t_0)) = \nu_{\rho_u}\left(\gamma^-, \gamma^+\right). \tag{8.3.6}$$

Furthermore, for all real number t_0 and t we have,

$$\begin{aligned}
&k_{t+t_\gamma}(u, (\gamma^-, \gamma^+, t_0)) \pi_2 \tilde{\sigma}(u, (\gamma^-, \gamma^+, t_0)) \\
&\quad = (k_t(u, (x, y, t_0)) + \alpha_{\rho_u}(\gamma)) \pi_2 \tilde{\sigma}(u, (\gamma^-, \gamma^+, t_0)).
\end{aligned}$$

Therefore we get that for all real number t_0

$$k_{t+t_\gamma}(u, (\gamma^-, \gamma^+, t_0)) = k_t(u, (\gamma^-, \gamma^+, t_0)) + \alpha_{\rho_u}(\gamma). \tag{8.3.7}$$

We also note that for all real number t and t' we have

$$\begin{aligned}
&k_{t+t'}(u, (x, y, t_0)) \pi_2 \tilde{\sigma}(u, (x, y, t_0)) \\
&\quad = k_t(u, (x, y, t_0 + t')) \pi_2 \tilde{\sigma}(u, (x, y, t_0 + t')) \\
&\qquad + k_{t'}(u, (x, y, t_0)) \pi_2 \tilde{\sigma}(u, (x, y, t_0)).
\end{aligned}$$

And using equation 8.3.5 we get that

$$k_{t+t'}(u, (x, y, t_0)) = k_t(u, (x, y, t_0 + t')) + k_{t'}(u, (x, y, t_0)). \tag{8.3.8}$$

Now we fix some real number $r > 0$ and define

$$\mathfrak{K}_t(u, (x, y, t_0)) := \log\left(\frac{\int_t^{r+t} \exp(k_s(u, (x, y, t_0))) ds}{\int_0^r \exp(k_s(u, (x, y, t_0))) ds}\right).$$

Using equation 8.3.7 we get that

$$\mathfrak{K}_{t+t_\gamma}(u, (\gamma^-, \gamma^+, t_0)) = \mathfrak{K}_t(u, (\gamma^-, \gamma^+, t_0)) + \alpha_{\rho_u}(\gamma). \tag{8.3.9}$$

Moreover, using equation 8.3.8 we get that

$$\mathfrak{K}_{t+t'}(u, (x, y, t_0)) = \mathfrak{K}_t(u, (x, y, t_0 + t')) + \mathfrak{K}_{t'}(u, (x, y, t_0)). \tag{8.3.10}$$

Finally we define

$$f_u(x, y, t_0) := \left.\frac{\partial}{\partial t}\right|_{t=0} \mathfrak{K}_t(u, (x, y, t_0)). \tag{8.3.11}$$

We notice that

$$\begin{aligned}
&\left.\frac{\partial}{\partial t}\right|_{t=0} \mathfrak{K}_t(u,(x,y,t_0)) = \left.\frac{\partial}{\partial t}\right|_{t=0} \log\left(\frac{\int_t^{r+t}\exp(k_s(u,(x,y,t_0)))ds}{\int_0^r \exp(k_s(u,(x,y,t_0)))ds}\right)\\
&= \left.\frac{\partial}{\partial t}\right|_{t=0} \log\left(\int_t^{r+t}\exp(k_s(u,(x,y,t_0)))ds\right)\\
&= \frac{\left.\frac{\partial}{\partial t}\right|_{t=0}\int_0^t \left(\exp(k_{s+r}(u,(x,y,t_0))) - \exp(k_s(u,(x,y,t_0)))\right)ds}{\int_0^r \exp(k_s(u,(x,y,t_0)))ds}\\
&= \frac{\exp(k_r(u,(x,y,t_0))) - \exp(k_0(u,(x,y,t_0)))}{\int_0^r \exp(k_s(u,(x,y,t_0)))ds}.
\end{aligned}$$

Therefore $f_u(x,y,t_0)$ is also μ-Hölder transeversely real analytic. Moreover, using equation 8.3.10 one gets

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \mathfrak{K}_t(u,(x,y,t_0)) = \left.\frac{\partial}{\partial t}\right|_{t=t_0} \mathfrak{K}_t(u,(x,y,0)).$$

Hence we have

$$\begin{aligned}
\int_0^{t_\gamma} f_u(\gamma^-,\gamma^+,s)ds &= \int_0^{t_\gamma} \left.\frac{\partial}{\partial t}\right|_{t=s} \mathfrak{K}_t(u,(\gamma^-,\gamma^+,0))ds \\
&= \mathfrak{K}_{t_\gamma}(u,(\gamma^-,\gamma^+,0)) - \mathfrak{K}_0(u,(\gamma^-,\gamma^+,0))\\
&= \alpha_{\rho_u}(\gamma).
\end{aligned} \tag{8.3.12}$$

Therefore if $u \in \mathcal{D}_0$ then

$$\int_\gamma f_u = \int_\gamma \mathfrak{f}_{\rho_u}$$

for all $\gamma \in \Gamma$. Now using theorem 7.1.4 we deduce that f_u is Liv̌sic cohomologous to the positive Hölder function $\mathfrak{f}_{\rho_u}$ for all $u \in \mathcal{D}_0$. Therefore for any flow invariant measure $\mathfrak{m}$ on $\mathsf{U}_0\Gamma$ we have

$$\int f_u d\mathfrak{m} = \int \mathfrak{f}_{\rho_u} d\mathfrak{m} > 0.$$

Now using lemma A.1 and lemma A.2 of [17] and transverse analyticity of f_u we derive that there exist a neighborhood $\mathcal{D}_1 \subset \mathcal{D}_0$ and there exist a real number $T > 0$ such that for all $u \in \mathcal{D}_1$

$$f_u^T(x,y,t_0) := \frac{1}{T}\int_0^T f_u(x,y,t_0+s)ds > 0.$$

Now we finish our proof by considering the collection

$$\{\mathsf{f}_u := f_u^T \mid u \in \mathcal{D}_1\}$$

and noticing that it satisfies all the required properties. □

8.4 Deformation of the cross ratio

In this section we obtain a formula for the variation of the cross ratio which is similar in taste to the theorem 3.4.3. We start by stating an alternative version of the proposition 10.4 from [8].

Proposition 8.4.1. *[Bridgeman, Canary, Labourie, Sambarino] Let ϱ be a point in $\mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$. Then*

$$\lim_{n\to\infty} \left(\ell_\varrho(\gamma^n\eta^n) - \ell_\varrho(\gamma^n) - \ell_\varrho(\eta^n)\right) = \log \mathsf{b}_\varrho(\eta^-, \gamma^-, \gamma^+, \eta^+)$$

where $\ell_\rho(\gamma)$ is the length of the closed geodesic corresponding to $\varrho(\gamma)$.

Lemma 8.4.2. *Let $\{\rho_t\}$ be a smooth path in $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$. Then the following holds*

$$\lim_{n\to\infty} \left.\frac{d}{dt}\right|_{t=0} \nu_{\rho_t}\left((\gamma^n\eta^n)^-, (\gamma^n\eta^n)^+\right) = \left.\frac{d}{dt}\right|_{t=0} \nu_{\rho_t}\left(\eta^-, \gamma^+\right).$$

Moreover, the rate of convergence is exponential.

Proof. As $\{\rho_t\}$ is a path in $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ we can consider it as a path in $\{\rho_u\}_{u\in\mathcal{D}}$, a complex analytic family in $\mathsf{Hom}(\Gamma, \mathsf{G}^{\mathbb{C}})$ parametrized by a complex disk $\mathcal{D}$ around 0. Using theorem 8.2.2 we get that the limit maps ξ^+ and ξ^- are μ-Hölder transversely complex analytic. Hence

$$\left\{\xi^+\left((\gamma^n\eta^n)^-\right) - \xi^+\left(\eta^-\right)\right\}_{n\in\mathbb{N}}$$

is a sequence of complex analytic maps converging to zero on $\mathcal{D}$. Moreover, as $(\gamma^n\eta^n)^-$ converges to η^- at an exponential rate and the limit map ξ^+ is μ-Hölder we get that the rate of convergence is exponential. Now as

$$\left\{\xi^+\left((\gamma^n\eta^n)^-\right) - \xi^+\left(\eta^-\right)\right\}_{n\in\mathbb{N}}$$

is a sequence of complex analytic functions on $\mathcal{D}$ converging exponentially to zero, using Cauchy's Integral formula we get that the derivative of the sequence is also converging exponentially to zero. Now restricting the limit maps on the real part we get that

$$\lim_{n\to\infty} \left.\frac{d}{dt}\right|_{t=0} \xi^+_{\rho_t}\left((\gamma^n\eta^n)^-\right) = \left.\frac{d}{dt}\right|_{t=0} \xi^+_{\rho_t}\left(\eta^-\right)$$

with the convergence rate being exponential. Similarly we get that

$$\lim_{n\to\infty} \left.\frac{d}{dt}\right|_{t=0} \xi^-_{\rho_t}\left((\gamma^n\eta^n)^+\right) = \left.\frac{d}{dt}\right|_{t=0} \xi^-_{\rho_t}\left(\gamma^+\right)$$

where the convergence rate is exponential.

Let $\tilde{\pi}_2$ be the projection from $\mathsf{U}\mathbb{A}$ onto S^1. We note that $\tilde{\pi}_2$ gives rise to a projection map

$$\pi_2 : \mathsf{N} \longrightarrow \mathsf{S}^1.$$

We conclude our proof by recalling from equation 8.3.6 that

$$\nu_\rho\left(\eta^-, \gamma^+\right) = \pi_2 \circ \left(\xi^+_\rho, \xi^-_\rho\right)\left(\eta^-, \gamma^+\right).$$

□

Proposition 8.4.3. *Let $\{\rho_t\}$ be a smooth path in $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$. Also let $X_{\rho_t(\gamma)}$ be any point on the unique affine line fixed by $\rho_t(\gamma)$ where γ is in Γ. Then for all γ, η in Γ we have*

$$
\begin{aligned}
&\lim_{n\to\infty} \left(\alpha_{\rho_t}(\gamma^n\eta^n) - \alpha_{\rho_t}(\gamma^n) - \alpha_{\rho_t}(\eta^n)\right) \\
&= \left\langle X_{\rho_t(\gamma)} - X_{\rho_t(\eta)} \mid \nu_{\rho_t}\left(\eta^-, \gamma^+\right) + \nu_{\rho_t}\left(\eta^+, \gamma^-\right)\right\rangle, \\
&\left.\frac{d}{dt}\right|_{t=0} \left\langle X_{\rho_t(\gamma)} - X_{\rho_t(\eta)} \mid \nu_{\rho_t}\left(\eta^-, \gamma^+\right) + \nu_{\rho_t}\left(\eta^+, \gamma^-\right)\right\rangle \\
&= \lim_{n\to\infty} \left.\frac{d}{dt}\right|_{t=0} \left(\alpha_{\rho_t}(\gamma^n\eta^n) - \alpha_{\rho_t}(\gamma^n) - \alpha_{\rho_t}(\eta^n)\right).
\end{aligned}
$$

Proof. We begin the proof by mentioning that the first identity is a variation of an identity worked out by Charette–Drumm in [11]. In fact I use the same method used by them to compute both the identities.

Let $l_{\rho(\eta)}$ be the unique affine line fixed by $\rho(\eta)$ and let $l^-_{\rho(\gamma)}$ be the affine plane parallel to the plane tangent to the null cone and containing $l_{\rho(\gamma)}$. As the space like affine lines $l_{\rho(\eta)}$ and $l_{\rho(\gamma)}$ are not parallel to each other we have that $l_{\rho(\eta)}$ intersects $l^-_{\rho(\gamma)}$ in a unique point Q_ρ. Also let R be the point on $l_{\rho(\gamma)}$ such that

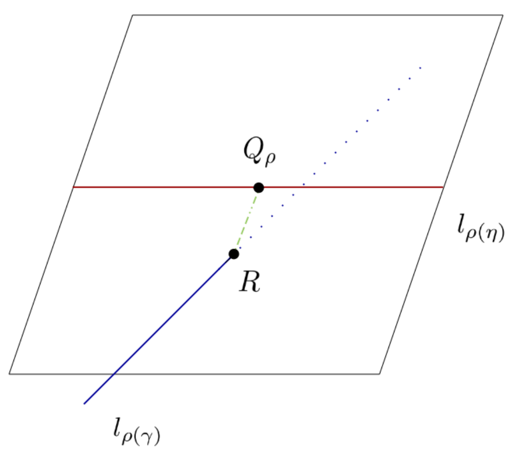
$$\langle R - Q_\rho, \nu_\rho(\gamma)\rangle = 0$$

where $\nu_\rho(\gamma) := \nu_\rho\left(\gamma^-, \gamma^+\right)$. We note that as $Q_\rho \in l_{\rho(\eta)}$ we have

$$Q_\rho - \rho(\eta)^{-n} Q_\rho = \alpha_\rho(\eta^n)\nu_\rho(\eta)$$

and as $R \in l_{\rho(\gamma)}$ we have

$$\rho(\gamma)^n R - R = \alpha_\rho(\gamma^n)\nu_\rho(\gamma).$$



Now we observe that

$$\begin{aligned}\alpha_\rho(\gamma^n\eta^n) &= \langle \rho(\gamma)^n Q_\rho - \rho(\eta)^{-n} Q_\rho \mid \nu_\rho(\gamma^n\eta^n)\rangle \\ &= \langle \rho(\gamma)^n Q_\rho - \rho(\gamma)^n R - (Q_\rho - R) \mid \nu_\rho(\gamma^n\eta^n)\rangle \\ &\quad + \langle (Q_\rho - \rho(\eta)^{-n} Q_\rho) + (\rho(\gamma)^n R - R) \mid \nu_\rho(\gamma^n\eta^n)\rangle \\ &= \langle (\mathsf{L}_\rho(\gamma)^n - \mathbb{I})\,(Q_\rho - R) \mid \nu_\rho(\gamma^n\eta^n)\rangle \\ &\quad + \langle \alpha_\rho(\gamma^n)\nu_\rho(\gamma) + \alpha_\rho(\eta^n)\nu_\rho(\eta) \mid \nu_\rho(\gamma^n\eta^n)\rangle.\end{aligned}$$

We observe that the vector $(Q_\rho - R)$ is an eigenvector of $\mathsf{L}_\rho(\gamma)$ with eigenvalue $\lambda_\rho(\gamma)$ such that $|\lambda_\rho(\gamma)| < 1$. Therefore we get that

$$\begin{aligned}\alpha_\rho(\gamma^n\eta^n) = &\ (\lambda_\rho(\gamma)^n - 1)\,\langle Q_\rho - R \mid \nu_\rho(\gamma^n\eta^n)\rangle \\ &+ \langle \alpha_\rho(\gamma^n)\nu_\rho(\gamma) + \alpha_\rho(\eta^n)\nu_\rho(\eta) \mid \nu_\rho(\gamma^n\eta^n)\rangle.\end{aligned}$$

We recall that

$$\left\langle \nu_\rho(\gamma) \mid \nu_\rho(\eta^-, \gamma^+)\right\rangle = 1 = \left\langle \nu_\rho(\eta) \mid \nu_\rho(\eta^-, \gamma^+)\right\rangle.$$

Hence we get

$$\begin{aligned}&\alpha_\rho(\gamma^n\eta^n) - \alpha_\rho(\gamma^n) - \alpha_\rho(\eta^n) \\ &= (\lambda_\rho(\gamma)^n - 1)\,\langle Q_\rho - R \mid \nu_\rho(\gamma^n\eta^n)\rangle \\ &\quad + \alpha_\rho(\gamma^n)\langle \nu_\rho(\gamma) \mid \nu_\rho(\gamma^n\eta^n) - \nu_\rho(\eta^-, \gamma^+)\rangle \\ &\quad + \alpha_\rho(\eta^n)\langle \nu_\rho(\eta) \mid \nu_\rho(\gamma^n\eta^n) - \nu_\rho(\eta^-, \gamma^+)\rangle.\end{aligned}$$

Now using the fact that $\nu_\rho(\gamma^n\eta^n)$ converges exponentially to $\nu_\rho(\eta^-, \gamma^+)$, while $\alpha_\rho(\gamma^n)$ has polynomial growth and the fact that $|\lambda_\rho(\gamma)| < 1$ we obtain

$$\lim_{n\to\infty} (\alpha_\rho(\gamma^n\eta^n) - \alpha_\rho(\gamma^n) - \alpha_\rho(\eta^n)) = -\langle Q_\rho - R \mid \nu_\rho(\eta^-, \gamma^+)\rangle.$$

Moreover, using lemma 8.4.2 and the fact that $|\lambda_\rho(\gamma)| < 1$ we deduce that

$$\begin{aligned}&\lim_{n\to\infty} \left.\frac{d}{dt}\right|_{t=0} (\alpha_{\rho_t}(\gamma^n\eta^n) - \alpha_{\rho_t}(\gamma^n) - \alpha_{\rho_t}(\eta^n)) \\ &= -\left.\frac{d}{dt}\right|_{t=0} \langle Q_{\rho_t} - R \mid \nu_{\rho_t}(\eta^-, \gamma^+)\rangle.\end{aligned}$$

Finally, we conclude by observing that

$$\langle R - Q_\rho \mid \nu_\rho(\eta^-, \gamma^+)\rangle = \langle X_{\rho(\gamma)} - X_{\rho(\eta)} \mid \nu_\rho(\eta^-, \gamma^+) + \nu_\rho(\eta^+, \gamma^-)\rangle$$

where $X_{\rho(\gamma)} \in l_{\rho(\gamma)}$ and $X_{\rho(\eta)} \in l_{\rho(\eta)}$ are any two points for $\gamma, \eta \in \Gamma$. □

Theorem 8.4.4. *Let $\{\varrho_t\}$ be a smooth path in $\mathsf{Hom}_{\mathrm{S}}(\Gamma, \mathsf{SO}^0(2,1))$ such that $\rho := (\varrho_0, \dot\varrho_0) \in \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ where $\dot\varrho_0 := \left.\frac{d}{dt}\right|_{t=0} \varrho_t$. Then we have*

$$\begin{aligned}&\langle X_{\rho(\gamma)} - X_{\rho(\eta)} \mid \nu_\rho(\eta^-, \gamma^+) + \nu_\rho(\eta^+, \gamma^-)\rangle \\ &= \left.\frac{d}{dt}\right|_{t=0} \log \mathsf{b}_{\varrho_t}(\eta^-, \gamma^-, \gamma^+, \eta^+)\end{aligned}$$

where $X_{\rho(\gamma)}$ is any point on the unique affine line fixed by $\rho(\gamma)$ and $X_{\rho(\eta)}$ is any point on the unique affine line fixed by $\rho(\eta)$.

Proof. The result follows from using theorem 3.4.3, proposition 8.4.1, lemma 8.4.2 and proposition 8.4.3. □

Chapter 9

Properties of the Pressure metric

9.1 The thermodynamic mapping

Let $\rho \in \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ and let $h_{\mathfrak{f}_\rho}$ be the topological entropy of the reparametrized flow on $\mathsf{U}_0\Gamma$ corresponding to the reparametrization $\mathfrak{f}_\rho$. By theorem 0.0.40 we know that the geodesic flow on $\mathsf{U}_{\mathrm{rec}}\mathsf{M}_\rho$ is metric Anosov. Hence by using proposition 7.2.1 we deduce that $h_{\mathfrak{f}_\rho}$ is finite and positive and moreover,

$$h_{\mathfrak{f}_\rho} = \lim_{T\to\infty} \frac{1}{T} \log\left(\#\left\{[\gamma] \in \mathsf{O}(\Gamma) \mid \int_\gamma \mathfrak{f}_\rho \leqslant T\right\}\right)$$

where $\mathsf{O}(\Gamma)$ is the set of closed orbits of $\mathsf{U}_0\Gamma$. We also recall that for all $\gamma \in \Gamma$

$$\int_\gamma \mathfrak{f}_\rho = \alpha_\rho(\gamma).$$

Therefore we see that $h_{\mathfrak{f}_\rho}$ only depends on the Livšic cohomology class of $\mathfrak{f}_\rho$. Hence we denote $h_{\mathfrak{f}_\rho}$ by h_ρ and we get that

$$h_\rho = \lim_{T\to\infty} \frac{1}{T} \log\left(\#\left\{[\gamma] \in \mathsf{O}(\Gamma) \mid \alpha_\rho(\gamma) \leqslant T\right\}\right). \tag{9.1.1}$$

Now using proposition 7.4.5 and proposition 8.3.1 we deduce that the map

$$\begin{aligned} h : \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G}) &\longrightarrow \mathbb{R} \\ \rho &\longmapsto h_\rho \end{aligned} \tag{9.1.2}$$

is analytic. We recall that the Gromov flow ψ on the compact metric space $\mathsf{U}_0\Gamma$ is Hölder. Now using lemma 7.1.3 and proposition 8.3.1 we deduce that the pressure of the map $-h_\rho \mathsf{f}_\rho$ is zero with respect to the Gromov flow ψ. Let $\mathcal{H}(\mathsf{U}_0\Gamma)$ be the set of all Livšic cohomology classes of pressure zero functions.

Definition 9.1.1. *We define the thermodynamic mapping as follows,*

$$\begin{aligned} \mathfrak{T} : \mathsf{Hom}(\Gamma, \mathsf{G}) &\longrightarrow \mathcal{H}(\mathsf{U}_0\Gamma) \\ \rho &\longmapsto [-h_\rho \mathsf{f}_\rho]. \end{aligned}$$

Lemma 9.1.2. *The map $\mathfrak{T}$ is analytic.*

Proof. The result follows from proposition 8.3.1 and the fact that the entropy funtion is also analytic. ☐

9.2 The Pressure metric

Let $I(f, g)$ be the *intersection number* of the two reparametrizations f and g. As our flow is metric Anosov, using theorem 7.2.3 and equation 7.1.1 we get that

$$I(\mathsf{f}_{\rho_1}, \mathsf{f}_{\rho_2}) = \lim_{T\to\infty} \frac{1}{\#R_T(\rho_1)} \sum_{[\gamma]\in R_T(\rho_1)} \frac{\alpha_{\rho_2}(\gamma)}{\alpha_{\rho_1}(\gamma)}$$

where $R_T(\rho_1) := \{[\gamma] \in \mathsf{O}(\Gamma) \mid \alpha_{\rho_1}(\gamma) \leqslant T\}$. And using proposition 7.4.5 and proposition 8.3.1 we notice that the map I is analytic. Let us define

$$J_{\rho_1}(\rho_2) := I(\rho_1, \rho_2)\frac{h_{\rho_2}}{h_{\rho_1}}.$$

Now using propositions 7.3.3, 7.4.1 and 7.4.4 we get the following result.

Proposition 9.2.1. *Let $\rho_1, \rho_2 \in \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$. Then $J_{\rho_1}(\rho_2) \geqslant 1$. Now if $J_{\rho_1}(\rho_2) = 1$ then there exist a positive real number c such that*

$$c\alpha_{\rho_1}(\gamma) = \alpha_{\rho_2}(\gamma)$$

for all $\gamma \in \Gamma$. Moreover, if $\{\rho_t\}$ is a smooth path in $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ then

$$\left.\frac{\partial^2}{\partial t^2}\right|_{t=0} J_{\rho_0}(\rho_t) = 0$$

if and only if $\left.\frac{d}{dt}\right|_{t=0} h_{\rho_t}\mathsf{f}_{\rho_t}$ is Livšic cohomologous to zero.

Definition 9.2.2. *Let $\rho \in \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ and let $v, w \in \mathsf{T}_\rho\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$. We define*

$$\mathtt{P}_\rho(v, w) := \mathsf{D}^2_\rho J_\rho(v, w).$$

The map $\mathtt{P}$ is called the pressure form on $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$.

Remark 9.2.3. *We notice that by proposition 9.2.1 the pressure form $\mathtt{P}$ on $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ is non-negative definite.*

9.3 Vectors with Pressure norm zero

In this section we will describe the zero vectors of the pressure norm.

Proposition 9.3.1. *Let $\{\rho_t\}$ be a smooth path in $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ with $\left.\frac{d}{dt}\right|_{t=0} \rho_t = v$. If $\mathtt{P}_\rho(v, v) = 0$ and $\left.\frac{d}{dt}\right|_{t=0} h_{\rho_t} = 0$ then for all γ in Γ*

$$\left.\frac{d}{dt}\right|_{t=0} \alpha_{\rho_t}(\gamma) = 0.$$

Proof. We start by using proposition 9.2.1 and notice that $\frac{d}{dt}\big|_{t=0} h_{\rho_t}\mathsf{f}_{\rho_t}$ is Livšic cohomologous to zero. Hence for all closed orbits $[\gamma] \in \mathsf{O}(\Gamma)$ we have that

$$\int_\gamma \frac{d}{dt}\bigg|_{t=0} h_{\rho_t}\mathsf{f}_{\rho_t} = 0.$$

Now we observe that

$$\begin{aligned} 0 &= \int_\gamma \frac{d}{dt}\bigg|_{t=0} h_{\rho_t}\mathsf{f}_{\rho_t} = \int_\gamma \left(\frac{d}{dt}\bigg|_{t=0} h_{\rho_t}\right)\mathsf{f}_{\rho_0} + \int_\gamma h_{\rho_0}\left(\frac{d}{dt}\bigg|_{t=0} \mathsf{f}_{\rho_t}\right) \\ &= h_{\rho_0}\int_\gamma \frac{d}{dt}\bigg|_{t=0} \mathsf{f}_{\rho_t} = h_{\rho_0}\frac{d}{dt}\bigg|_{t=0}\int_\gamma \mathsf{f}_{\rho_t} = h_{\rho_0}\frac{d}{dt}\bigg|_{t=0} \alpha_{\rho_t}(\gamma). \end{aligned}$$

We conclude by recalling that the entropy h_{ρ_0} is positive and hence our result follows. □

Lemma 9.3.2. *If for all $\gamma \in \Gamma$ we have $\frac{d}{dt}\big|_{t=0} \alpha_{\rho_t}(\gamma) = 0$ then for all $\gamma, \eta \in \Gamma$ we have*

$$\frac{d}{dt}\bigg|_{t=0} \mathsf{b}_{\rho_t}\left(\eta^+, \gamma^-, \gamma^+, \eta^-\right) = 0.$$

Proof. Using proposition 8.4.3 we get that

$$\frac{d}{dt}\bigg|_{t=0} \left\langle X_{\rho_t(\gamma)} - X_{\rho_t(\eta)} \mid \nu_{\rho_t}\left(\eta^-, \gamma^+\right) + \nu_{\rho_t}\left(\eta^+, \gamma^-\right)\right\rangle = 0$$

and also

$$\frac{d}{dt}\bigg|_{t=0} \left\langle X_{\rho_t(\gamma)} - X_{\rho_t(\eta)} \mid \nu_{\rho_t}\left(\eta^+, \gamma^+\right) + \nu_{\rho_t}\left(\eta^-, \gamma^-\right)\right\rangle = 0.$$

Now using identities 2.4.5, 2.4.7 and 2.4.9 we get that

$$\begin{aligned} &\mathsf{b}_{\rho_t}\left(\eta^+, \gamma^-, \gamma^+, \eta^-\right)\left(\nu_{\rho_t}\left(\eta^+, \gamma^+\right) + \nu_{\rho_t}\left(\eta^-, \gamma^-\right)\right) \\ &= \mathsf{b}_{\rho_t}\left(\eta^-, \gamma^-, \gamma^+, \eta^+\right)\left(\nu_{\rho_t}\left(\eta^-, \gamma^+\right) + \nu_{\rho_t}\left(\eta^+, \gamma^-\right)\right) \\ &= \left(1 - \mathsf{b}_{\rho_t}\left(\eta^+, \gamma^-, \gamma^+, \eta^-\right)\right)\left(\nu_{\rho_t}\left(\eta^-, \gamma^+\right) + \nu_{\rho_t}\left(\eta^+, \gamma^-\right)\right). \end{aligned}$$

Therefore we deduce that

$$\frac{d}{dt}\bigg|_{t=0} \mathsf{b}_{\rho_t}\left(\eta^+, \gamma^-, \gamma^+, \eta^-\right) = 0$$

for all $\gamma, \eta \in \Gamma$. □

Proposition 9.3.3. *Let $\{\rho_t\}$ be a smooth path in $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ with $\frac{d}{dt}\big|_{t=0} \rho_t = \dot\rho_0$. If $\mathsf{P}_{\rho_0}(\dot\rho_0, \dot\rho_0) = 0$ and $\frac{d}{dt}\big|_{t=0} h_{\rho_t} = 0$ then*

$$[\dot\rho_0] = 0$$

in $\mathsf{H}^1_{\rho_0}(\Gamma, \mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of the Lie group G and $\mathsf{H}^1_{\rho_0}(\Gamma, \mathfrak{g})$ is the group cohomology.

Proof. Using proposition 9.3.1 and lemma 9.3.2 we get that

$$\left.\frac{d}{dt}\right|_{t=0} \mathsf{b}_{\rho_t}\left(\eta^+, \gamma^-, \gamma^+, \eta^-\right) = 0$$

for all $\gamma, \eta \in \Gamma$. Now using proposition 10.1 of [8] we deduce that

$$\left[\left.\frac{d}{dt}\right|_{t=0} \mathtt{L}_{\rho_t}\right] = 0$$

in $\mathsf{H}^1_{\mathtt{L}_{\rho_0}}(\Gamma, \mathfrak{so}(2,1))$. Therefore without loss of generality we can take

$$\mathtt{L}_{\rho_t} = \mathtt{L}_{\rho_0}$$

for all t. Now again using proposition 9.3.1 we get that

$$\left.\frac{d}{dt}\right|_{t=0} \left\langle \mathtt{u}_{\rho_t}(\gamma) \mid \nu_{\rho_t}\left(\gamma^-, \gamma^+\right)\right\rangle = 0$$

for all $\gamma \in \Gamma$. We notice that ν_ρ only depends on $\mathtt{L}_\rho$. Therefore

$$\nu_{\rho_t} = \nu_{\rho_0}$$

for all t and we obtain

$$\left\langle \left.\frac{d}{dt}\right|_{t=0} \mathtt{u}_{\rho_t}(\gamma) \mid \nu_{\rho_0}\left(\gamma^-, \gamma^+\right)\right\rangle = 0$$

for all $\gamma \in \Gamma$. Now using theorem 1.2 of [11] we deduce that

$$\left[\left.\frac{d}{dt}\right|_{t=0} \mathtt{u}_{\rho_t}\right] = 0$$

in $\mathsf{H}^1_{\mathtt{L}_{\rho_0}}(\Gamma, \mathfrak{so}(2,1))$. Hence it follows that

$$\left[\left.\frac{d}{dt}\right|_{t=0} (\mathtt{L}_{\rho_t}, \mathtt{u}_{\rho_t})\right] = [\dot{\rho}_0] = 0$$

in $\mathsf{H}^1_{\rho_0}(\Gamma, \mathfrak{g})$. □

9.4 Margulis Multiverse

Let h_ρ be the topological entropy related to a representation $\rho \in \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$. We recall from equation 9.1.1 that

$$h_\rho = \lim_{T\to\infty} \frac{1}{T} \log\left(\#\left\{[\gamma] \in \mathsf{O}(\Gamma) \mid \alpha_\rho(\gamma) \leqslant T\right\}\right). \tag{9.4.1}$$

Moreover, we also recall that the map

$$\begin{aligned} h : \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G}) &\longrightarrow \mathbb{R} \\ \rho &\longmapsto h_\rho \end{aligned} \tag{9.4.2}$$

is analytic. Now we define the *constant entropy sections* of $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ for any positive real number k as follows:

$$\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_k := \{\rho \in \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G}) \mid h_\rho = k\} . \tag{9.4.3}$$

We note that if $(\varrho, \mathtt{u})$ is in $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3) = \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ then so is $(\varrho, c\mathtt{u})$ where c is some positive real number.

Lemma 9.4.1. *Let* $(\varrho, \mathtt{u})$ *be in* $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3)$ *then for any positive real number* c *we have*

$$h_{(\varrho, c\mathtt{u})} = \frac{1}{c}\, h_{(\varrho, \mathtt{u})} .$$

Proof. Using the definition of the Margulis invariant we have that

$$\begin{aligned}
\alpha_{(\varrho, c\mathtt{u})}(\gamma) &= \langle c\mathtt{u}(\gamma) \mid \nu_\varrho(\gamma)\rangle \\
&= c\,\langle \mathtt{u}(\gamma) \mid \nu_\varrho(\gamma)\rangle = c\,\alpha_{(\varrho, \mathtt{u})}(\gamma) .
\end{aligned}$$

where $\nu_\varrho(\gamma) := \nu_\varrho(\gamma^-, \gamma^+)$. Now using equation 9.4.1 we get that

$$\begin{aligned}
h_{(\varrho, c\mathtt{u})} &= \lim_{T\to\infty} \frac{1}{T} \log\left(\#\left\{\gamma \in \Gamma \mid \alpha_{(\varrho, c\mathtt{u})}(\gamma) \leqslant T\right\}\right) \\
&= \lim_{T\to\infty} \frac{1}{T} \log\left(\#\left\{\gamma \in \Gamma \mid \alpha_{(\varrho, \mathtt{u})}(\gamma) \leqslant \frac{T}{c}\right\}\right) \\
&= \frac{1}{c} \lim_{T\to\infty} \frac{1}{T} \log\left(\#\left\{\gamma \in \Gamma \mid \alpha_{(\varrho, \mathtt{u})}(\gamma) \leqslant T\right\}\right) = \frac{1}{c}\, h_{(\varrho, \mathtt{u})} .
\end{aligned}$$

□

Proposition 9.4.2. *Let* $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_k$ *be a constant entropy section for some real number* k *then* $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_k$ *is a codimension one analytic submanifold of* $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$.

Proof. We consider the analytic map h and using lemma 9.4.1 notice that

$$\left.\frac{d}{dt}\right|_{t=0} h\left(\varrho, \frac{1}{1+t}\mathtt{u}\right) = h(\varrho, \mathtt{u}) \left.\frac{d}{dt}\right|_{t=0} (1+t) \neq 0.$$

Hence $\mathtt{Rk}(\mathsf{D}_{(\varrho,\mathtt{u})} h) = 1$. Now using the Implicit function theorem we conclude that $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_k = h^{-1}(k)$ is an analytic submanifold of $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ with codimension 1. □

Remark 9.4.3. *The following map*

$$\begin{aligned}
\mathcal{I}_k : \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_1 &\longrightarrow \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_k \\
(\varrho, \mathtt{u}) &\longmapsto \left(\varrho, \frac{1}{k}\mathtt{u}\right)
\end{aligned}$$

gives an analytic isomorphism between $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_1$ *and* $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_k$.

Proposition 9.4.4. *The space* $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ *is analytically isomorphic to the product space* $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_1 \times \mathbb{R}$.

Proof. We define two analytic maps as follows

$$\begin{aligned} \mathfrak{h} : \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G}) &\longrightarrow \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_1 \times \mathbb{R} \\ \rho = (\varrho, \mathtt{u}) &\longmapsto (\varrho, h_\rho \mathtt{u}) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{h}' : \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_1 \times \mathbb{R} &\longrightarrow \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G}) \\ ((\varrho, \mathtt{u}), r) &\longmapsto \left(\varrho, \frac{1}{r}\mathtt{u}\right). \end{aligned}$$

We conclude our result by observing that $\mathfrak{h}' \circ \mathfrak{h} = \mathsf{Id}$ and $\mathfrak{h} \circ \mathfrak{h}' = \mathsf{Id}$. □

Definition 9.4.5. *We define the Margulis multiverse with entropy k to be*

$$\mathcal{M}_k := \mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})_k/\sim$$

where k is a positive real number and $\rho_1 \sim \rho_2$ if and only if ρ_1 is a conjugate of ρ_2 by some element of the group $\mathsf{G} = \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$.

9.5 Riemannian metric on Margulis Multiverse

In this section we finally prove that the pressure metric $\mathtt{P}$ restricted to the constant entropy sections of $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ is Riemannian.

Proof of Theorem 0.0.42. We consider the definition 9.4.5 and observe that the result follows from proposition 9.3.3 and proposition 9.4.2. □

Proof of Theorem 0.0.43. Let $\rho = (\mathtt{L}_\rho, \mathtt{u}_\rho)$ be a point in $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$ and for $\epsilon > 0$ let

$$\{\rho_t := (\mathtt{L}_\rho, (1+t)\mathtt{u}_\rho)\}_{t \in (-\epsilon, \epsilon)}$$

be a smooth path in $\mathsf{Hom}_{\mathrm{M}}(\Gamma, \mathsf{G})$. We notice that if f_0 is a reparametrization coming from ρ then

$$\mathsf{f}_t := (1+t)\mathsf{f}_0$$

is a reparametrization which comes from ρ_t. We also notice that the entropy

$$h_{\rho_t} = \frac{h_\rho}{1+t}.$$

Therefore we get

$$\left.\frac{d}{dt}\right|_{t=0} h_{\rho_t}\mathsf{f}_{\rho_t} = \left.\frac{d}{dt}\right|_{t=0} h_\rho \mathsf{f}_\rho = 0.$$

Hence by proposition 9.2.1 we get that $\mathtt{P}(\dot\rho_0, \dot\rho_0) = 0$ where $\dot\rho_0 := \left.\frac{d}{dt}\right|_{t=0} \rho_t$ and $[\dot\rho_0] \neq 0$ in $\mathsf{H}^1_{\rho_0}(\Gamma, \mathfrak{g})$. Now using remark 9.2.3 we conclude that $\mathtt{P}$ has signature $(\dim(\mathcal{M}) - 1, 0)$ over the moduli space $\mathcal{M}$. □

Bibliography

[1] ABELS, H.: *"Properly discontinuous groups of affine transformations, A survey"* Geometriae Dedicata 87, 309-333, (2001).

[2] ABELS, H., MARGULIS, G. A. and SOIFER, G.: *"Properly discontinuous groups of affine transformations with orthogonal linear part."* C. R. Acad. Sci. Paris Sr. I Math. 324 (3), 253-258, (1997).

[3] ABELS, H., MARGULIS, G. A. and SOIFER, G.: *"On the Zariski closure of the linear part of a properly discontinuous group of affine transformations."* J. Diff. Geo. 60 (2), 315-344, (2002).

[4] BONAHON, F.: *"The geometry of Teichmüller space via geodesic currents,"* Invent. Math. 92, 139-162, (1988).

[5] BOWEN,R.: *"Symbolic dynamics for hyperbolic flows"* Amer. J. Math. 95 (1973), 429-460.

[6] BOWEN, R. and RUELLE, D.: *"The ergodic theory of axiom A flows"* Invent. Math. 29 (1975), 181-202.

[7] BRIDGEMAN, M.: *"Hausdorff dimension and the Weil-Petersson extension to quasifuchsian space"* Geom. and Top. 14 (2010), 799-831.

[8] BRIDGEMAN, M., CANARY, R., LABOURIE, F. and SAMBARINO, A.: *"The pressure metric for convex homomorphisms."* arXiv preprint arXiv:1301.7459 (2013).

[9] BRIDGEMAN, M. and TAYLOR, E.: *"An extension of the Weil-Petersson metric to quasi-Fuchsian space"* Math. Ann. 341 (2008), 927-943.

[10] CHAMPETIER, C.: *"Petite simplification dans les groupes hyperboliques,"* Ann. Fac. Sci. Toulouse Math. 3, 161221, (1994).

[11] CHARETTE, V. and DRUMM, T.: *"Strong marked isospectrality of affine Lorentzian groups."* J. Differential Geom. 66 , no. 3, 437–452, (2004) .

[12] CHARETTE, V., GOLDMAN, W. M. and JONES, C. A.: *"Recurrent Geodesics in Flat Lorentz 3-Manifolds."* (2001).

[13] DANCIGER, J, GUÉRITAUD, F. and KASSEL, F.: *"Geometry and Topology of Complete Lorentz SpaceTimes of Constant Curvature."* (2013).

[14] DRUMM, T.: *"Fundamental polyhedra for Margulis space-times"*, Doctoral Dissertation, University of Maryland (1990).

[15] FRIED, D. and GOLDMAN, W. M.: *"Three-dimensional affine crystallographic groups."* Adv. in Math. 47, 1-49, (1983).

[16] GOLDMAN, W. M. : *"The Margulis Invariant of Isometric Actions on Minkowski (2+1)-Space".* Rigidity in Dynamics and Geometry, pp 187-201, (2002).

[17] GOLDMAN, W. M. and LABOURIE, F.: *"Geodesics in Margulis spacetimes."* Ergodic Theory and Dynamical Systems, 32, 643-651, (2012).

[18] GOLDMAN, W. M., LABOURIE, F. and MARGULIS, G.: *"Proper affine actions and geodesic flows of hyperbolic surfaces."* Annals of mathematics 170.3, 1051-1083, (2009).

[19] GOLDMAN, W. M. and MARGULIS, G.: *"Flat Lorentz 3-manifolds and cocompact Fuchsian groups,"* Contemporary Mathematics, 262, pp 135-146, (2000).

[20] GROMOV, M.: *"Hyperbolic groups,"* in Essays in Group Theory, MSRI Publ. 8, 75263, (1987).

[21] GUICHARD, O. and WIENHARD, A.: *"Anosov representations: Domains of discontinuity and applications."* Inventiones Mathematicae, Volume 190, Issue 2, 357-438, (2012).

[22] HIRSCH, M. W., PUGH, C. C. and SHUB, M.: *"Invariant manifolds, Lecture Notes in Mathematics"* Vol. 583, (1977).

[23] HUBBARD, J.: *"Teichmüler theory and applications to geometry, topology and dynamics."* Vol 1., Matrix Editions, Ithaca.

[24] LABOURIE, F.: *"Fuchsian affine actions of surface groups."* J. Diff. Geo. 59 (1), 15 - 31, (2001).

[25] LABOURIE, F.: *"Anosov Flows, Surface Groups and Curves in Projective Space."* Inventiones Mathematicae, Volume 165, Issue 1, 51-114, (2006).

[26] LIVSIC, A.N.: *"Cohomology of dynamical systems"* Math. USSR Izvestija 6 (1972).

[27] MARGULIS, G. A.: *"Free completely discontinuous groups of affine transformations."* Dokl. Akad. Nauk SSSR 272, 785-788, (1983).

[28] MARGULIS, G. A.: *"Complete affine locally flat manifolds with a free fundamental group."* J. Soviet Math. 134, 129-134, (1987).

[29] MCMULLEN, C.: *"Thermodynamics, dimension and the Weil-Petersson metric"* Invent. Math. 173 (2008), 365-425.

[30] MESS, G.: *"Lorentz spacetimes of constant curvature."* Geom. Dedicata 126, 3-45, (2007).

[31] MILNOR.J: *"On fundamental groups of complete affinely flat manifolds."* Adv. in Math.25, 178-187, (1977).

[32] MINEYEV, I.: *"Flows and joins of metric spaces,"* Geom. Top. 9, 403482, (2005).

[33] PARRY, W. and POLLICOTT, M.: *"Zeta functions and the periodic orbit structure of hyperbolic dynamics"* Astérisque 187-188 (1990).

[34] POLLICOTT, M.: *"Symbolic dynamics for Smale flows"* Am. J. of Math. 109 (1987), 183-200.

[35] RUELLE, D.: *"Thermodynamic Formalism"* Addison-Wesley, London (1989).

[36] SAMBARINO, A.: *"Quantitative properties of convex representations,"* Comm. Math. Helv., 89, 443-488 (2014).

[37] SHUB, M.: *"Global Stability of Dynamical Systems"* Springer-Verlag, (1987).

[38] WOLPERT, S.: *"Thurstons Riemannian metric for Teichmüller space"* J. Diff. Geom. 23 (1986), 143-174.